COMMUNICATION NETWORKS

VOL. II

THE CLASSICAL THEORY OF LINES, CABLES AND FILTERS

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COMMUNICATION NETWORKS

VOL. I

THE CLASSICAL THEORY OF LUMPED CONSTANT NETWORKS

BY
ERNST A. GUILLEMIN, Ph.D.
Professor of Electrical Communications,
Massachusetts Institute of Technology

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PREFACE

In writing the present volume, I have attempted to fill a need for an introductory treatment of classical network analysis which would be thorough enough to give the reader more than just an elementary knowledge of circuit theory. I believe that the more immediate use for a deeper understanding of network principles lies in the field of communications. Wherever illustrative examples are used, I have, therefore, placed the emphasis upon communication networks.

The existing texts on communication principles deal with the theory and practical significance of special types of networks commonly used in radio or telephone circuits. Progressive research, however, demands a more thorough grasp of the fundamental methods of attack on network problems in general. The sources where such information may be found are at present widely scattered throughout the literature on mathematics and mechanics, as well as in various advanced treatises on circuit theory. Besides being scattered, these sources are rather difficult to read because they naturally presuppose a general understanding of the subject. The student who is beginning the study of network theory, therefore, finds it difficult to piece together a coherent and consistent picture out of such material as this.

In the text I have not limited myself to the consideration of such information which has a direct bearing upon practical forms. The most abstract ideas have received as much space as the more applicable ones. This point of view is as essential in elementary courses as it is in advanced study; and it applies to technical as well as purely scientific curricula.

Regarding the presentation of the subject, I have deliberately chosen what seems offhand to be an illogical order. I have started with some general remarks regarding mathematical methods and analogies; and instead of proceeding with a logical presentation of fundamental principles, I have spent two chapters on the application of network principles to the single mesh circuit. This is done for the purpose of acquainting the reader with the general subject. Without this acquaintance he would be unable to appreciate the reasons for the procedure in the general case. Generalization is meaningless unless one has some idea of what is being generalized. Perhaps I should have spent more time on the discussion of special cases before starting with the methods of attack for the general case. I have assumed, however, that the reader would naturally supplement this "warming-up stage" with other material, either in the classroom or laboratory. The consideration of the general case I have attempted to put into as logical a form as possible.

In the chapter on the evaluation of integration constants, I have given — besides the usual method — what I call the direct method. I do not believe that this method is generally known among engineers. The analytic form looks very formidable, but I hope that this will not disturb the reader too much.

The chapter on the vector interpretation of the transient solution is given merely for the purpose of throwing additional light upon the natural behavior of the system. The vector method employed is well known in the study of linear transformations, but not generally known in network analysis.

The subject of complex notation I have placed as prominently in the foreground as possible. Its importance cannot be overemphasized in this kind of work because it obviates the necessity of operating with the clumsy sines and cosines and their explicit phase angles. These matters are particularly emphasized when Fourier series enter into the problem. Here the complex form is almost a necessity.

The chapter on Heaviside's and the superposition formulae, I have made very brief because these more special matters are ably treated by other authors. I have included them here, however, to point out their relation to the classical theory and to give the student a working knowledge of them.

The last chapter on the Fourier representation of a periodic force function is included chiefly for the purpose of introducing the reader to the complex form and its uses. The presentation proceeds from the usual sine and cosine form, instead of from an independent derivation, because I assumed that this method of approach would be more easily understood by engineers. I have given a number of illustrations which show how much more convenient the complex form is in connection with network problems. The treatment of the microphone and condenser transmitter circuits is added in order to illustrate how the complex form is used for the solution of circuits in which the parameters are periodic functions of time.

It may be well to say a few words here regarding the use of so-called advanced methods of attack in a book which is intended primarily as an introductory treatment. Methods are frequently designated as advanced merely because they are not in current use. To the student, the entire field is new; the advanced methods are no exception. If they afford a better understanding of the situation involved, then it is good pedagogy to introduce them into an elementary discussion. It is well for the teacher to bear in mind that methods which are very familiar to him are not necessarily the easiest for the student to grasp.

In conclusion, I wish to state that I am chiefly indebted to Professor Arnold Sommerfeld, through whose teaching I gained my first real insight into the philosophy of oscillatory systems, and particularly in the use of complex methods as compared to the trigonometric representation. I am also most grateful to the Electrical Engineering Department of the Massachusetts Institute of Technology, through whose coöperative attitude this work has been made possible. In particular, I wish to express my thanks to Professor E. L. Bowles and Messrs. E. A. Johnson and F. M. Gager for suggestions and criticisms regarding the arrangement of material. Acknowledgment is also due to my wife, Sallie, who typed all of the copy and otherwise helped prepare the manuscript.

E. A. G.

Cambridge, Mass.
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INTRODUCTION

Before going into the subject matter of our present topic, it will be well to get some idea of what our general problem is, and how it differs from that of electrical engineering in other fields, as well as of the manner in which the material will be presented. Basically, communication networks are no different from any other kind except that they frequently are more complicated than most circuits met with in power work. Not only this, but they usually involve a larger number of variable factors and conditions of excitation. The elementary methods of treating networks are able to cope with only the simplest cases, and much is left to guess work and experimentation that can be done by more thorough analysis. It rapidly becomes clear to the beginner that actual circuits used in the laboratory bear only a general resemblance to the idealized cases treated in his text book. This situation is bad enough at present, but is becoming more aggravated daily. Clearly it is time that we studied more powerful methods of attack, and introduced methods of analysis that enable a deeper insight into the fundamental relationships which govern the behavior of the kinds of systems we have to deal with.

The student of electrical communications should early recognize the importance of methods of representation, of analysis and synthesis which will compress and unify a large number of otherwise separate ideas. Nothing is complicated in the sense that the individual steps involved in any thought process are difficult to grasp. The complication lies in the fact that the particular problem considered involves such a large number of individual steps which cannot be separately appreciated until a vision of the entire process is attained. The average mind cannot hold many ideas simultaneously. Yet it must do so in order to fully appreciate the significance of the entire problem. Not only must this one problem be grasped, but it must be cataloged and identified with respect to a larger class to which it belongs.

The answer to this seemingly impossible state of affairs lies in more advanced methods of analysis. By this we mean methods which will compress a series of elementary operations or thought processes into one operation. The medium by which we are able to do this is one which engineers have called upon for help as long as the profession has existed. We refer to the science of mathematics. Mathematics is nothing more than a kind of shorthand plus some rules for manipulating the symbols involved. Whole sentences and paragraphs are compressed into a single line by means of mathematical symbols. We often allude to this characteristic by the term of "Mathematical Language." The only difference between elementary mathematics and advanced mathematics is that the latter is still shorter than the former. In other words, advanced mathematical methods have a higher power of compressing ideas. This is exactly what we need to alleviate the dilemma that we are getting into in the case of communication networks.

The present treatment of network theory is intended to be elementary in the sense that it presupposes no particular knowledge of any portion of this field. In the presentation of the subject, however, we shall make use of more advanced methods wherever it seems advisable to do so. Advanced methods as such are no harder for the student to learn than so-called elementary ones. Any student of a new subject must master new ideas, and he may as well spend his time mastering something that will be of infinitely more use to him later on.

One outstanding difference does exist between elementary and advanced analysis from the pedagogical standpoint, and that is that certain elementary mathematical rules and operations can be learned by pure memorizing, whereas advanced methods can be gotten only by good hard thinking. This thinking has to be done by the student — the text book cannot do that for him.

In this book we shall endeavor to present methods by which certain situations may be attacked. No formulae will be given by means of which the answers to specific problems may be directly obtained. These the student is expected to work out for himself as the occasion arises. Our object is simply to show what means are available for carrying out the logical analysis of a given network problem which will eventually lead to its solution.

The word "solution" is one which we may as well define before going any farther. Many students have the idea that the solution to a given network means the determination of the current or charge in some specific part of the network as a function of the parameters involved and the independent variable or variables — time, distance, etc. This is only one form that a solution might take. Frequently it is next to impossible, from practical considerations, to find such an explicit solution. Often it is not even necessary. In general, any information, which can be obtained by any manner whatsoever, that will aid in throwing light upon the mechanism or behavior of the system under consideration, contributes to a better understanding of the factors which govern the phenomenon and therefore constitutes a more or less complete solution. Solutions are hardly ever complete in the sense that they tell us all that can possibly be known about the circuit we are studying. The term " complete solution " is, of course, often used in mathematical terminology and has a definite meaning when applied to a specific operation. But in the sense of gaining a complete knowledge of the network, the complete solution is rarely found.

In the following chapters various theorems and methods of analysis will be studied with this one objective in mind, namely that by means of these the student will be able to see the operation of the network from a new angle. This in fact is the main feature or object of the book — to present as many ways of looking at the general network problem as possible. For this reason much of the material may even seem dry and uninteresting to the reader who has not had the experience of this method before. Any new subject will seem uninteresting until sufficient material has been mastered in order to appreciate its value. This is especially true of advanced methods of analysis. At first the presentation seems far removed from the main interest of the problem which is being treated. Pedagogically this is unfortunate because it is liable to discourage the student before he gets to that point where he will be rewarded for his patience and effort. But if he will only persevere through this " desert stage," he will be amply rewarded.

These matters are particularly stressed in these opening paragraphs because this book is intended for engineers who have as a rule not been used to this sort of thing. The writer is an engineer himself, and speaks from personal experience. Engineers are not satisfied with a " proof " unless they can visualize every individual step and see the reason for doing the particular thing involved. This is a laudable characteristic, but it can be overworked. In this connection the writer is reminded of an anecdote which is ascribed to the French mathematician, Laplace. A student of his had been struggling with various theorems and proofs, but did not see why some of the steps were justified, and in general was in a fog as to just where all this was going to lead him. The professor tried hard to enlighten him, but rather found that he was entangling the poor fellow all the more. Finally he ended the interview by saying: " Go on young man, go on, conviction will come to you later! " Thus it is frequently impossible to see the " drift " of an argument until the whole presentation is complete, and it is unreasonable of the student to expect anything else.

If the reader will bear these matters in mind while studying the following chapters, he will save himself much unnecessary uneasiness. If he will carry on even though temporarily confused, he will find that the whole will become clearer later on. It may very well be that much of the material is unnecessary for a superficial understanding of simple circuits. But the communication engineer cannot afford to be content with just this much. Network theory is his stronghold — or should be. It is therefore fitting and proper that he should be introduced to this subject in a manner which is entirely comparable in the matter of thoroughness and insight with the importance which it has in his chosen line of work.

COMMUNICATION NETWORKS

CHAPTER I

THE PHILOSOPHY OF LINEAR ELECTRICAL NETWORKS

1. Fundamental Mechanical Analogies. The engineer, like the physicist, likes to be able to visualize the mechanism of his investigations. In fact he is so insistent upon this point that he is frequently willing to depart quite materially from the subject in hand and create artificial pictures for his mental satisfaction when direct visualization seems hopeless. Thus, for example, when he treats a problem involving numerous parameters, he frequently calls them " dimensions " and talks about a " space " for these " dimensions " to exist in. Having created this " n-dimensional space," he proceeds to call an n-1 dimensional thing in it an " n-dimensional plane " — etc. To the beginner this may seem to confuse things rather than to clarify them. But in doing the above, we are correlating our new ideas with those of geometry, with which we are usually more familiar, and thus we bring the new ideas closer to known ground.

Another and perhaps still more valuable aid toward visualizing mechanisms — particularly in the case of network theory — is through the medium of analogies to mechanical systems. For example, in mechanics we have mass — any material object has mass. We know that the force acting upon an object equals its mass times the acceleration which results from the application of the force when no friction is present. Thus:

$$F = ma. \tag{1}$$

But if we call distance x, then velocity becomes $v = \frac{dx}{dt}$, and acceleration becomes $\frac{dv}{dt} = \frac{d^2x}{dt^2}$, so that:

$$F = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}. \tag{2}$$

Now in electricity we know that if we apply an electromotive force or a voltage E to a coil of wire having an inductance of L henries and negligible resistance, the equation of dynamic equilibrium becomes:

$$E = L\frac{di}{dt} \tag{3}$$

where i is the current that flows in the coil. If we remember that an electric current is due to the flow of charge, then we recognize that $i = \frac{dq}{dt}$ (q being the charge in practical coulombs for i in practical amperes). Hence:

$$E = L\frac{d^2q}{dt^2}. \tag{4}$$

Thus we can already draw several analogies between the mechanical and electrical systems. An e.m.f. or voltage is like a force; inductance is like mass; charge is like distance or displacement; current is like velocity. Symbolically we have:

$$\left.\begin{array}{c} F \backsim E \\ m \backsim L \\ x \backsim q \\ v \backsim i \end{array}\right\}. \tag{5}$$

There is nothing rigorous about this, and the above presentation makes no claim as to rigor. It is foolish to talk about rigor when making or constructing analogies. They are made only to help the mind retain ideas, and fix ideas by the well-known method of correlation.

Consider a spring having no internal friction, fastened at one end, and with no influence from gravity upon it. If we push the spring at the free end, it will push back just as much. If we do not exceed the elastic limit, the pushing force will be equal to a constant times the displacement, or the distance the end has been pushed. Symbolically:

$$F = Kx. \tag{6}$$

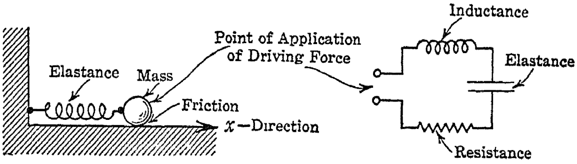
We call K the elastance of the spring. It is also called other things, but we shall call it the elastance. Now let us suppose on the other hand that we have an electrical capacitance or a so-called condenser before us and apply a potential difference to its terminals. To every e.m.f. applied there corresponds a certain amount of electrical charge which is forced into the condenser. The relation is a linear one. The higher the voltage, the larger the charge becomes. We have symbolically:

$$E = \frac{q}{C}. \tag{7}$$

Comparing this situation with that of the spring, we see that since $q \backsim x$ and $E \backsim F$, it follows that $1/C \backsim K$. In order to bring this idea closer we write for the electrical case:

$$S = \frac{1}{C} \tag{8}$$

and call S the elastance of the condenser. As a matter of fact, it is usually preferable to use S instead of C as will be seen later.



(*a*) (*b*)

FIG. 1

Finally we should consider frictional forces. This matter hardly needs emphasis. In mechanics we have:

$$F = fv \tag{9}$$

while in the electrical case:

$$E = Ri. \tag{10}$$

Hence the coefficient of dynamic friction f is analogous to resistance R. Of course, the mechanical case is not always as simple as expressed by (9) except for a limited range of velocity, but this need not disturb our analogy.

Putting all this together we see that a mechanical system having mass, elastance, and friction, is like an electrical circuit having inductance, capacity (or elastance), and resistance. This is illustrated in Fig. 1 (*a*) and (*b*).

Let us follow this analogy still farther. Suppose the ball in the mechanical system runs in a groove so that it is constrained to move in the x-direction as indicated. The applied force will also be considered in the x-direction. Then our condition for dynamic equilibrium would be:

$$m\frac{dv}{dt} + fv + Kx = F. \tag{11}$$

This equation says that the sum of the counter-forces of inertia, friction, and elasticity must at every instant be counterbalanced by the impressed force. This equation may be written in either of the following forms in terms of displacement or velocity alone:

$$m\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx = F \tag{11a}$$

$$m\frac{dv}{dt} + fv + K\int v\,dt = F. \tag{11b}$$

For the electrical circuit we have the corresponding condition of dynamic equilibrium for an applied e.m.f. in the circuit:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + Sq = E \tag{12a}$$

or:

$$L\frac{di}{dt} + Ri + S\int i\,dt = E. \tag{12b}$$

The electrical equations state that the sum of electrical counter voltages around the circuit must equal the impressed voltage at every instant of time.

In mechanics it is usual to write this equation of dynamic equilibrium in terms of displacement x, as given by (11a). In electricity on the other hand, the equation (12b) in terms of current is the common form. From fundamental considerations, and also for the sake of analogy, we should write our electric circuit equations in terms of charge (as in 12a) rather than current. It is not always convenient to do this for other reasons however, but it should be remembered that the charge equation is really the fundamental one. The electrical mesh-charge is analogous to the mechanical displacement in the x-direction or the x-coördinate-displacement. We begin to see already that not only are charge and displacement analogous, but the electrical mesh seems to be the analog of the mechanical coördinate or dimension. We shall get a clearer picture of this later on.

2. Mechanical Degrees of Freedom and their Network Analog. The systems just discussed are fully described by means of a single quantity or coördinate — distance in the x-direction in the mechanical case, and charge in the electrical case. We say that these systems have only one **degree of freedom.** The number of degrees of freedom of a mechanical system or of an electrical system is equal to the *least* number of variables it takes to specify the condition of the system. If we know x as a function of time in the mechanical system, or q as a function of time in the electrical system, then we are able to answer any question about the particular system we are interested in. For instance in the case of the electrical system we can determine the current at any time by simply differentiating the expression for charge with respect to time. Knowing the current, we can calculate the electromagnetic energy. From the charge we also know the electrostatic energy as a function of time. All these things are known at every instant of time by knowing just one thing to start with. The same is true of the mechanical system. This is what we mean by the system having one degree of freedom. Another way of looking at this is to say that the mechanical system has only one degree of freedom because it can move in one direction only, namely in the x-direction. For the electrical system we can say in general that there is only one mesh and hence only one degree of freedom.

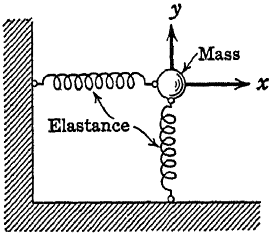


FIG. 2

Let us proceed to more complicated systems. Consider the mechanical system shown in Fig. 2. Here we have a ball resting on a plane surface, (xy-plane) held by two springs. The ball may now move in two directions, x and y. More than that — a displacement in the x-direction causes a pull in the y-direction and vice versa. We say that the x- and y-directions are elastically coupled. It now takes two parameters or coördinates to specify the position of the ball at any instant. Hence we say that the system has two degrees of freedom. The analogous electrical system is shown in Fig. 3. Here we have two meshes coupled by an elastance. The resistances are the analogs of table-top friction in the x- and y-directions respectively. We must now have two mesh-charges or currents to specify this electrical system. These two correspond to the two coördinates x and y in the mechanical case. The electrical system of Fig. 3 therefore possesses two degrees of freedom like the mechanical system of Fig. 2, and these two are elastically coupled in both cases. Note that the two cases are not equivalent, but only analogous. The difference lies merely in a question of rigor which we are not going into here. We are merely trying to illustrate the fact that the network analog to a mechanical dimension or coördinate is a mesh.

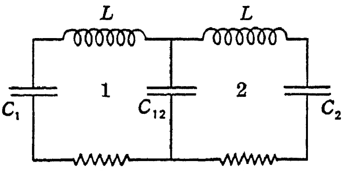


FIG. 3

Thus a three-mesh network is analogous to a mechanical system having three dimensions or coördinates for its specification. These three meshes may be coupled in various ways by means of inductance, elastance, and resistance. Mechanically the motion in the three directions may be coupled in various ways by means of springs and constraints such as wires for the ball to slide on, etc. We shall not go into this latter point because it would take up too much time, and is relatively unimportant for our present purpose. These electrical and mechanical systems would be said to possess three degrees of freedom because it would take three things to completely specify the condition in which they are at any instant.

Electrically we can easily see how the idea continues. Each additional mesh adds a coördinate or dimension in the mechanical sense. This may require a few remarks on the mechanical side. Here we can also have as many dimensions as we please. Take for example, the mechanical system shown in Fig. 4. Here we have three mass points held together by springs which are fastened

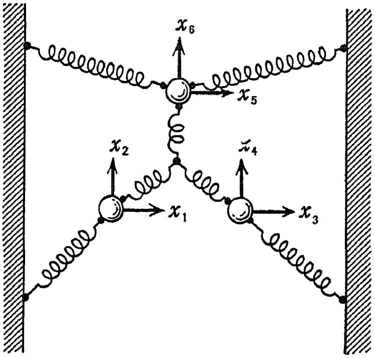


FIG. 4

at the right and left. The balls themselves rest upon a plane surface. Since it takes two coördinates to specify the position of each ball, it will take six coördinates to specify the entire system at any time. In the figure these are designated by $x_1, x_2, \ldots x_6$. We often say that this is a six-dimensional problem. It has six degrees of freedom. The network analog would be a six-mesh network requiring six mesh-charges or mesh-currents to completely specify it.

The reader may follow out some of these ideas more fully for himself for the sake of practice. The above illustrates the principles involved.

3. Analogous Energy Relations. Other analogies between mechanical and electrical systems may be derived from energy relations. In the mechanical case we know that the kinetic energy of a mass point is given by:

$$K.E. = \tfrac{1}{2}\, mv^2. \tag{13}$$

The electromagnetic energy stored in the inductance on the other hand is:

$$M.E. = \tfrac{1}{2}\, Li^2. \tag{14}$$

Thus kinetic and electromagnetic energies are analogous. We often speak of the kinetic energy of an electric circuit meaning its electromagnetic energy.

In a similar way the potential energy stored in a compressed spring and that stored in a charged condenser are analogous. For these we have:

$$\left.\begin{array}{l}\text{Mech. P.E.} = \tfrac{1}{2}\, Kx^2 \\ \text{Elect. P.E.} = \tfrac{1}{2}\, Sq^2\end{array}\right\}. \tag{15}$$

Moreover for the mechanical system the time-rate of frictional energy loss is given by:

$$\text{Mech. Loss} = f\left(\frac{dx}{dt}\right)^2 = fv^2 \tag{16}$$

while for the electrical case we have:

$$\text{Elect. Loss} = R\left(\frac{dq}{dt}\right)^2 = Ri^2. \tag{17}$$

The student should note that these energy expressions (eqs. 13 to

17) are all quadratic. This is an important characteristic to which we shall return later.

4. Kinds of Motion. We shall leave this discussion of analogies between mechanical and electrical systems for the time being to say a few things about motion in general. There are two principal kinds of motion. One kind stays in the same direction all the time, although its rate of change may vary considerably. This kind is called **monotonic** and is illustrated in Fig. 5. The other kind is **oscillatory,** which means that the direction of motion changes back and forth as shown in Fig. 6. Most things in nature are oscillatory — the variation of temperature, of light intensity, of the tides, etc. with time, for example, are oscillatory "motions." In the case of communication networks, oscillatory motion is by far the most important.

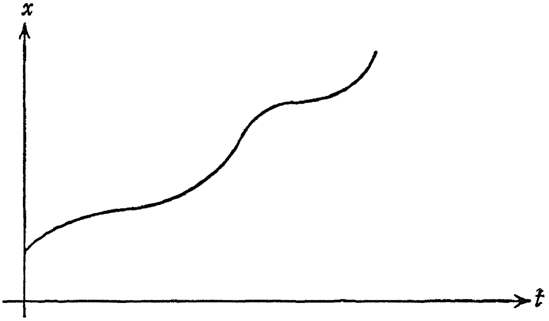


Fig. 5

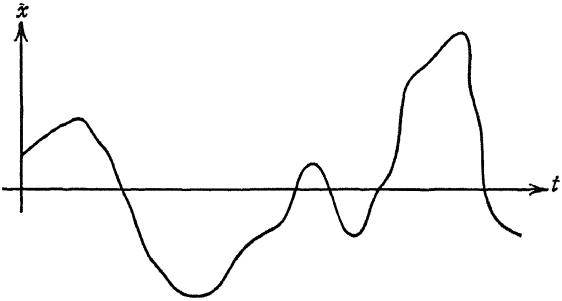


Fig. 6

It should be understood here that oscillatory motion is not necessarily **periodic.** Periodic motion is a particular kind of oscillatory motion. In order for a given motion to be periodic it must be possible to find a definite interval of time (or whatever the independent variable is) such that for the succeeding interval of same duration the function exactly repeats itself. This kind of motion is also called **cyclic** because it repeats after an accurately definable cycle. Motions which are not strictly periodic but nearly so are referred to as **quasi-periodic** or **almost-periodic.** Very frequently quasi-periodic motions are treated as periodic with sufficient accuracy for engineering purposes.

Periodic or cyclic motion can be further specialized. The particular kind of periodic motion defined by the sine or cosine functions is very common in electrical work and is referred to as **harmonic.** Thus motion in general can be classified as follows:

- **Monotonic**
- **Oscillatory**
  - Periodic or cyclic
    - Harmonic

Various combinations of the above.

5. Mathematical Methods of Attack. We come now to some important mathematical considerations in regard to the fundamental methods by which the solution to electrical networks is attacked. The essential point in this connection is to develop a thorough understanding of the so-called linear differential equation with constant coefficients.

Eqs. (11), (11*a*), (11*b*), (12*a*) and (12*b*) are of this type. They are of the second order. Consider the following form of this equation:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t). \tag{18}$$

In our former illustration the $f(t)$ was a constant. This equation is called **inhomogeneous** because the terms do not all involve the same variable. It would be homogeneous in x except for the right-hand side which is an arbitrary function of time. If we remove this $f(t)$, the equation becomes **homogeneous,** thus:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0. \tag{19}$$

This is called the **corresponding homogeneous** equation to eq. (18).

Let us discuss the method of solving (18). This solution consists of a function of time which will satisfy the equation. This function is called an **integral** of (18). By an integral is meant that function of time which substituted for x in each term of (18) on the left-hand side, will make that side equal to $f(t)$, so that the equation becomes an identity.

The form of this integral will obviously depend upon the form of $f(t)$. It will be an integral so chosen as to particularly fit the force-function $f(t)$. This integral is therefore called the **particular integral.** We do not proceed to find it until $f(t)$ is given.

Now in network theory there is practically only one kind of function which always appears in the place of $f(t)$; and in our work we shall consider only this one. It will be shown later how almost any other kind of function met with in network theory can be reduced to this one. This function is the **exponential** function:

$$f(t) = Ae^{mt} \tag{20}$$

where A and m are any constants and e is the base of the natural system of logarithms, or the so-called **Napierian base.** We therefore wish to find a function of time which will satisfy the equation:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = Ae^{mt}. \tag{18a}$$

We shall now introduce a method of solution which is very probably new to the reader, but one which is used almost exclusively by physicists and engineers. This method is nothing more than judicious guessing. The more elegant title by which this method is known is the **heuristic method.** What is meant is to guess at a function that seems most reasonable from inspection of the form of $f(t)$. Since in our case this is an exponential function, the reasonable thing to assume is that the particular integral is also an exponential function. In applying the heuristic method it is also reasonable to try the simplest assumption first. Hence we shall assume that the exponential function which satisfies (18a) has the same exponent as the right-hand side. We will merely write it with a different amplitude, thus:

$$x = Be^{mt}. \tag{21}$$

Substituting this into each term of the left-hand side of (18*a*) we have:

$$(am^2 + bm + c)Be^{mt} = Ae^{mt}. \tag{22}$$

Apparently the heuristic method did not work, because the left-hand side of (22) does not look much like the right-hand side. But let us see if we cannot make it so. In fact, this is quite simple. The only restriction imposed upon B is that it shall be a constant. Now since a, b, c, and m are constants by definition, we see that if we make:

$$B = \frac{A}{am^2 + bm + c} \tag{23}$$

then B will be a constant, and the eq. (22) will become an identity. We therefore have the desired solution by substituting the value of B given by (23) into (21), thus:

$$x = \frac{Ae^{mt}}{am^2 + bm + c}. \tag{24}$$

This is all there is to the heuristic method. It may not seem like a straight forward method at first, but from the standpoint of directness and elegance, it cannot be beaten. Of course, if $f(t)$ is more complicated it may not be possible to hit upon the correct assumption the first time. But with a little practice and familiarity with the nature of various functions it will soon become apparent after a few trials what form the solution should take. As far as network problems are concerned, the reader need not worry about this point anyway, because it hardly ever occurs that $f(t)$ is other than an exponential function. As a matter of fact, the exponential function can almost be called the natural function. Practically everything that occurs in nature varies as an exponential function in some way or another. It has been said — in a rather humorous way to be sure — that the only mathematics that a physicist need know is the exponential function. The writer might add that if the electrical engineer will only learn to know the characteristics and the behavior of the exponential function from all possible angles, he will have mastered nine-tenths of the mathematics that he needs for his purposes.

Let us continue with the discussion of our inhomogeneous linear differential equation with constant coefficients. Eq. (18*a*) for which we have just found the particular integral, is of the second order. The method however is exactly the same for any order. Suppose we have:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdot \cdot \cdot + a_1 \frac{dx}{dt} + a_0 x = Ae^{mt}. \qquad (25)$$

We again assume (21) as the solution. Substituting into (25) we get:

$$(a_n m^n + a_{n-1} m^{n-1} + \cdot \cdot \cdot + a_1 m + a_0) Be^{mt} = Ae^{mt} \qquad (26)$$

from which it is apparent that:

$$B = \frac{A}{a_n m^n + a_{n-1} m^{n-1} + \cdot \cdot \cdot + a_1 m + a_0} \qquad (27)$$

and again we have the solution. Hence it is equally simple to find the particular integral regardless of the order of the differential equation.

As mentioned in a previous paragraph, this particular integral that we have just found depends for its character and form upon the kind of function that appears on the right-hand side of the differential equation. This function is usually called the **force-function.** The nature of the particular integral is intimately tied up with that of the force-function. In a way it imitates or reflects the behavior of that function in a manner which is tempered or influenced to some extent by the make-up of the left-hand side of the equation. It is through a recognition of these characteristics that the heuristic method of approach operates. The full significance of the method will however be appreciated after we couple these ideas with the physical considerations involved in the actual problem whose equilibrium condition is expressed by the given equation. This will be gone into more thoroughly in a succeeding paragraph.

We are for the time being considering the solution to the linear differential equation given in (18) from the mathematical standpoint only. Following along these lines we stop to consider for a moment what relation the corresponding homogeneous eq. (19) might have to the general solution of (18). Suppose we found a function of time that was an integral of (19). This would be such a function which substituted into (19) would reduce it to an identity. Hence it would reduce the left-hand side of (18) to zero. If we added this integral of (19) to the particular integral of (18) found previously, then this sum would reduce the left-hand side of (18) to $f(t)$ plus zero. This would still be an integral of (18). It would be a more general integral. Eq. (19) may possess several integrals, and their sum could be added to the particular integral of (18) to form a still more general solution of (18).

The sum of the integrals of (19) is called the **complementary function,** and is part of the **complete** or **general solution** of (18). This complementary function will now be found by using the heuristic method again. Since the exponential function turned out to be an integral of (18a), we will try it again in the case of (19). The difference now is that we have no idea of what to use in the exponent. Let us therefore assume any symbol at all and see whether we can determine its value later. Hence we write for an integral to (19):

$$x = Xe^{pt}. \tag{28}$$

Substituting this into (19) we get:

$$(ap^2 + bp + c)\,Xe^{pt} = 0. \tag{29}$$

If (28) is an integral of (19), then (29) must be an identity, i.e., the left-hand side of (29) must be zero. This might be done in several ways. One is to put X or e^{pt} equal to zero. This would be perfectly correct from the mathematical standpoint. It would lead to:

$$x = 0$$

as a solution of (19), which is quite obviously correct, but is also obviously useless. We call solutions like this **trivial solutions.** They are perfectly correct in all mathematical respects, but of no value. What we want is a **non-trivial** solution. Therefore there is just one possibility left for us to do in order to make (29) an identity, and that is to demand that:

$$ap^2 + bp + c = 0. \tag{30}$$

If this fails then (28) cannot be an integral of (19). However, since a, b, and c are known constants, (30) can be satisfied by fixing the value of p properly, and this is allowable because p has not yet been fixed. As a matter of fact, we are quite happy to find out what p is.

Now (30) is simply a quadratic equation with p as the variable. We know that it has two roots, i.e., there are two values of p that will satisfy (30). We therefore see that we can write two integrals of the form of (28) which will satisfy (19). Hence their sum is the desired complementary function. It is:

$$x = X_1e^{p_1t} + X_2e^{p_2t} \tag{31}$$

where p_1 and p_2 are the roots of (30), and X_1 and X_2 are any constants. Thus the complete solution of (18a) finally is:

$$x = \frac{Ae^{mt}}{am^2 + bm + c} + X_1e^{p_1t} + X_2e^{p_2t}. \tag{32}$$

Let us discuss this solution a bit before going on with other matters. We have merely stated that X_1 and X_2 are any constants. The reader might ask how their values are finally determined. This is a rather long story and will be gone into in detail later. However, we may say here that mathematically at least they represent the constants of integration. Eq. (18a) is of the second order. Hence it is logical that we should arrive at a solution involving two integration constants because any straight forward integration process of solution would have involved two integrations. It was by means of our heuristic method that we arrived at the solution (32) more directly, but unless it contained the requisite number of integration constants, it could not have been correct. This is logical if we consider that the solution (32) represents a physical action of some kind and this must be unique unless the conditions allow more than one possible result. In terms of physical networks we know without proof that a given circuit with given excitation will always behave in the same manner.

In order to complete this discussion we should show how the complementary function for the general case given by (25) is found. First of all the corresponding homogeneous equation to (25) is:

$$a_n\frac{d^nx}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0x = 0. \tag{33}$$

For its solution we assume:

$$x = Xe^{pt}$$

just as in the previous simpler case. Substituting into (33) we get:

$$(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0) X e^{pt} = 0. \tag{34}$$

Again we argue that if we want non-trivial solutions we must demand that:

$$a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0 = 0. \tag{35}$$

This is an algebraic equation of the nth degree and hence possesses n roots. If we denote these by $p_1, p_2, \ldots p_n$, then the complementary function for this case becomes:

$$x = X_1 e^{p_1 t} + X_2 e^{p_2 t} + \cdots + X_n e^{p_n t}. \tag{36}$$

The complete solution to (25) is therefore (21) plus (36). This solution contains n integration constants $X_1, \ldots X_n$ as it should because the given equation was of the nth order.

The eq. (35), or (30) for the simpler case, is called the **determinantal equation** because it determines the values of p in the complementary function. Its degree is always equal to the order of the given differential equation.

6. The Physical Side of the Mathematical Method. Now let us approach this subject again from the physical angle, i.e., from the standpoint of what all this means in terms of what happens in the network. This requires a further study of the equilibrium conditions in the network. We will go into this now, and use for our purpose the simple circuit shown in Fig. 7.

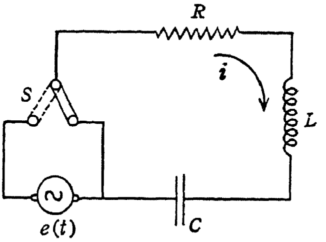


FIG. 7

This figure represents a resistance, an inductance, and a capacitance connected in series and so arranged by means of the switch S that an alternator, having a terminal voltage defined by $e(t)$, may be thrown into the circuit. We will suppose that this switch is thrown from the given position to the dotted position at a time $t = 0$. We will assume further that the voltage function is fixed by the relation:

$$e(t) = Ee^{j\omega t}. \tag{37}$$

It will be shown later that this expression represents a harmonic function with an angular frequency of ω radians per second or an actual frequency of $\frac{\omega}{2\pi}$ cycles per second. This function $e(t)$ is also referred to as the force actuating the circuit, as the driving force, or as the force-function.

The equilibrium condition after $t = 0$ must express the fact that the counter voltages in the circuit equal the impressed voltage at all instants; or that the counter-forces or e.m.f.'s equal the impressed force for all values of time. This is **Kirchhoff's e.m.f. law** applied to instantaneous values. In symbolical language we have:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = Ee^{j\omega t}. \tag{38}$$

This is an inhomogeneous linear differential equation with constant coefficients such as was given by (18a). Let us now see what the solution of this equation means physically. We know from the discussion in an earlier part of this chapter that the mechanical analogy to this circuit is given by a ball at the end of a spring as shown in Fig. 1(a). The force-function in this case is harmonic. The question that we are asking then is: How will the ball move if we apply a force to it that attempts to push it back and forth in a harmonic manner and at an angular frequency of ω radians per second? The electrical circuit asks the same question relative to the motion of charge, since current is merely the rate at which electrical charge moves. If we were asked this question outright we would probably answer that we could not say off-hand just how the motion would start out, but that it was reasonable to suppose that eventually the ball would move back and forth very much like the driving force wanted it to, namely with harmonic motion, and at an angular frequency of ω radians per second. Consequently it would be logical to assume that:

$$i = Ie^{j\omega t}. \tag{39}$$

This is the physical side of the heuristic method of solution. It is the common sense method. If we substitute (39) into (38), we have:

$$\left(Lj\omega + R + \frac{1}{Cj\omega}\right)Ie^{j\omega t} = Ee^{j\omega t}$$

from which it is quite evident that:

$$I = \frac{E}{R + j\left(L\omega - \dfrac{1}{C\omega}\right)}. \tag{40}$$

Hence the particular integral in this case is:

$$i = \frac{Ee^{j\omega t}}{R + j\left(L\omega - \dfrac{1}{C\omega}\right)}. \tag{40a}$$

But this we recognize as an old friend. It is the steady-state current which flows in the circuit, and is equal to the voltage divided by the steady-state impedance in complex form. Thus the particular integral turns out to be the **steady-state** performance of the network.

This is what we should expect. The steady state is an expression of how the network will eventually behave, and our heuristic method was guided intuitively by this eventual behavior.

The question now arises: Does this steady-state performance start immediately upon closing the switch? Let us arrive at the answer to this question in the following way. At the instant the switch S is thrown so that the generator is suddenly inserted into the circuit, the latter may not be at rest, but may still be oscillating from some previous shock. It may not necessarily be oscillating, but at least it will be behaving in a manner peculiar to itself — consistent with its own natural tendencies. At this instant the applied force suddenly demands that this natural behavior be stopped and a new regime put in its place. The writer likes to compare this situation with its human counterpart as follows:

Suppose we had a company of soldiers in a military camp somewhere who were under no supervision, and hence were having a good time doing just as they pleased about everything. Now suppose that suddenly a military commander were to appear upon the scene who had very definite ideas about how a company of soldiers should act, and that he put the camp under his control. Would the company immediately fall into step and act according to the commander's wishes in every detail? Certainly not. Eventually they would — assuming perfect discipline of course. But for some time the camp would be in a more or less chaotic condition. The result at first would be due to the interaction of two tendencies — one, the desire to obey orders as imposed by the commander, and the other an expression of the natural tendencies of the soldiers, their individual and group habits. There would be a transient period during which these natural tendencies were gradually overcome. The important point is that the initial stages would be modified by the natural mode of behavior. This natural mode is important in the determination of the transient behavior.

These characteristics can be translated almost without modification to the case of electrical circuit behavior. During the initial cycles following the switching operation, the behavior will be due to an interaction of the driving force, and the forces of natural expression. More accurately, the behavior during the transient period will be a linear superposition of the steady state and the natural behaviors of the circuit.

Hence it will be necessary to determine the natural behavior. But this corresponds to the behavior under the condition of no impressed forces — or the so-called **force-free** behavior. It is not hard to see that for our case this must be given by the solution of the following equation:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0. \tag{41}$$

This is the corresponding homogeneous equation to (38). Its solution is what we previously called the complementary function. We shall now call it the **transient portion** of the complete solution, or briefly the **transient solution.**

In applying the heuristic method to the homogeneous eq. (19), it could be characterized as hardly more than a pure guess method. There was no physical background to serve as a guide in forming our guess. The situation in connection with eq. (41) is now considerably different. This equation expresses physically the equilibrium condition of an actual circuit which is isolated from the influence of any form of excitation whatever. We know from experience that this system, like its mechanical analog — the ball on a spring —, is capable of performing certain motions when left alone after having been subjected to a shock which threw it out of its position of stationary equilibrium. The solution to (41) which we are seeking is therefore nothing else than this natural mode of motion of the system, or, to be more exact, the natural mode of current variation in our electrical circuit of Fig. 7.

Now we also know from experience that almost any kind of current variation that takes place in an electrical circuit may be represented by an exponential function of time. It may of course take several exponential terms to describe the exact behavior in many cases, but essentially the motion is still said to be exponential. If we assume for instance that the solution to (41) is given by:

$$i = Ae^{pt} \tag{42}$$

then we make no pretence of knowing anything as yet about either A or p. A is eventually going to fix the amplitude or the value of this current term at the initial instant $t = 0$. It will be shown in more detail later how the initial time phase of the current term is also contained in this amplitude A by virtue of its complex character. The constant p on the other hand has a more important rôle to play. Suppose that it were a real number. Then the function (42) would either increase or decrease in magnitude with time depending upon whether p were positive or negative respectively. If p were purely imaginary, the right-hand side of (42) would represent the sum of a sine and a cosine term having angular frequencies equal to the magnitude of p. This would represent oscillatory motion of a constant amplitude. If, lastly, p were a complex number, then the function given by (42) would represent the product of two terms — one an exponentially increasing or decreasing term depending upon whether the real portion of p were positive or negative respectively, and the other a sum of a sine and cosine having a frequency (angular frequency) equal to the imaginary portion of p. The result would be an oscillation which increases or decreases in amplitude with time. By means of several terms of this kind, any oscillatory variation of current can be represented. From these considerations therefore there is hardly any doubt that (42), or perhaps a sum of such terms, will suffice to describe the natural behavior of networks. Thus guided by reasoning, the heuristic method becomes the logical one by which to attack the solution to differential equations describing the behavior of physical systems.

In writing (42) we are merely trying to fulfill the demands of eq. (41). We must also remember that it is our privilege to adjust the values of A and p in any way we please in order to make (42) an integral of (41). With this in mind we substitute (42) into (41) and get:

$$\left(Lp + R + \frac{1}{Cp}\right)Ae^{pt} = 0. \tag{43}$$

If (42) is to be an integral of (41), then this eq. (43) must be an identity, i.e., the left-hand side must be identically equal to zero for all values of time. Another way of looking at this situation is to say that (43) is a **condition equation.** It says in effect that (42) will be an integral of (41) provided (43) can be satisfied.

One obvious conclusion to which we arrive from inspecting this condition (43) is to say that A must be zero. This would certainly fulfill all the requirements. Thus we would arrive at $i = 0$ for the solution to (41). If we try this in (41) we see that it does turn out to be an integral in the sense that it reduces (41) to the identity $0 \equiv 0$. What does this mean physically? It means that the rest condition is a natural condition for our circuit. It means that if the circuit is at rest, and is subjected to no external forces of excitation, then it will remain at rest forever. This is however not exactly what we wished to know. We really wanted to know how the circuit would act if left to itself in a condition of unrest, just as, for example, it would be after it had been disturbed by some force and then let go while still in action. The mechanical analogy would be to give the ball at the end of a spring a push and then let go of it.

Hence we must inspect (43) again and see whether it is possible to satisfy this condition in another way except that of making $A = 0$. Thus we arrive at the conclusion that we must demand that:

$$Lp + R + \frac{1}{Cp} = 0. \tag{44}$$

This is a further condition equation. If (44) can be satisfied by means of a constant value of p, then (42) using this value of p becomes in fact an integral of (41). But (44) is equivalent to the following quadratic equation in p:

$$p^2 + \frac{R}{L}p + \frac{1}{LC} = 0. \tag{44a}$$

By the fundamental law of algebra this equation can in general be satisfied by two different values of p. These are called the roots of (44a) or of (44). Each will give rise to an integral of the form of (42). If each of these integrals separately reduces the left-hand side of (41) to zero, then their sum will reduce it to zero plus zero. Hence it becomes clear that the complete transient solution is given by:

$$i = A_1 e^{p_1 t} + A_2 e^{p_2 t} \tag{45}$$

where p_1 and p_2 are the roots of (44) or (44a), and A_1 and A_2 are arbitrary and different from zero.

The roots of the determinantal equation (44) or (44a) have now a physical significance. From what was said in a previous paragraph, they govern the character of the variation of current with time — whether it be monotonic, or oscillatory, or both at once, and in the oscillatory case fix the frequency of oscillation. These roots p_1 and p_2 therefore express or control the character of the natural behavior of the circuit. They contain in effect the natural rates of growth or decay, as well as the natural frequencies of the system which they characterize. They are, therefore, variously known by such names as **natural angular frequencies, natural angular velocities, complex angular velocities, natural decrements and frequencies, natural modes,** etc. Since these roots are sometimes purely real and sometimes complex, they describe either the decrement alone or the frequency and corresponding decrement together. The significance of these remarks will be more fully appreciated later after a better understanding of the whole matter is obtained.

To summarize, we have found from these considerations that the physical network significance of the particular integral to our linear inhomogeneous differential equation with constant coefficients is the steady-state response of that network to the continued application of an arbitrary driving force, the latter being the right-hand member of the equation, or what we previously called the force-function. We further recognized the corresponding homogeneous equation as an expression of dynamic equilibrium under force-free conditions, as evidenced by the absence of any force-function on the right-hand side. The complementary function, which is the mathematical name for the solution to this homogeneous equation, is physically a description of the natural behavior of the network. In network theory we call it the transient solution. The roots of the determinantal equation are the natural modes of decay and of oscillation. The complete solution is a linear superposition of the forced and the natural behavior, i.e., of the steady state and the transient solutions of the system. For our example of Fig. 7 this becomes:

$$i = \frac{Ee^{j\omega t}}{R + j\left(L\omega - \frac{1}{C\omega}\right)} + A_1e^{p_1t} + A_2e^{p_2t}. \tag{46}$$

7. Integration Constants and their Relation to Initial Conditions. We come now to a very interesting phase of our present discussion, namely the subject of integration constants. In eq. (46) these are given by the constants A_1 and A_2 in the transient solution. The steady-state solution is complete in itself and contains no arbitrary constants. This is always the case. It is the object of the following paragraphs to show why this should be so, and what the real utility of integration constants is.

Students usually regard integration constants as a nuisance that has to be tolerated in network problems. From the standpoint of mechanical routine in performing the necessary motions involved in arriving at a numerical result, this is certainly correct. But from the physical standpoint, the function of integration constants is quite otherwise.

Suppose we consider again our mechanical analogy of a ball at the end of a spring. We have a driving force which is oscillatory in character and capable of taking hold of the ball at any instant we please to designate. The ball may further be still in motion — presumably from some previous disturbance — at the instant our driving force takes hold. It would be absurd to suppose that the exact behavior of the ball at any instant subsequent to the taking-hold of the driving force would be identical regardless of whether the ball was moving or not, or how it was moving at the instant of engagement with the force, or of the direction and magnitude of the latter at the same instant. To visualize this point, suppose for example that at the instant of engagement, the ball and the force were both moving in the same direction. The result would be comparatively free from shock and the initial behavior relatively undisturbed. If the ball and the force were traveling in opposite directions on the other hand, it is clear that there would be a considerable shock at the instant of engagement, and, although the eventual steady state would be the same in both cases, the transient period would be quite different. The difference between the two cases lies only in the different conditions under which engagement of the force took place. These conditions are referred to as the **initial conditions** of the system and of the driving force or the source of excitation.

Now suppose that we were required to write down some mathematical functions of time which were to describe the two resultant motions considered above. Assuming that we accomplished this, would the two expressions look alike? They would certainly have to differ somewhere. Now suppose that we have set up the differential equation for the dynamic equilibrium of the system we have just described. This equation, it will be remembered, expresses the equilibrium condition at every instant for any instantaneous value of the driving force or of the position or motion of the ball. Its complete solution must be ready to meet any situation at the instant of engagement. If this solution were complete in itself, i.e., if there were no unknown constants in it, then it would be a perfectly definite function of time. How could it represent the two different results written down above? How could it meet the infinite number of conditions which are possible to exist at the initial instant? The answer is that it could not meet these conditions, and hence would be absolutely useless. In fact it would be no solution at all.

The reason for this is that no information as to initial conditions is contained in the differential equation, and hence it is unreasonable to expect that the solution to this equation should give a definite answer as to the result. It must necessarily be incomplete, but at the same time it must possess sufficient flexibility to shape itself according to the demands of the initial conditions. A formal mathematical solution — such as eq. (46) — is like raw material. Solving a differential equation is like filling an order for raw material according to prescribed specifications. Before the material becomes useful in the way of a finished product, it must be hewn and molded into a particular shape as required by the peculiarities of the job. Unless the raw material possesses the required flexibility to allow this molding and chopping, it is useless. In the same way a mathematical solution is subjected to a molding process which fits it to the particular job it has to perform. The formal solution must possess as many degrees of flexibility as the number of arbitrary initial or other conditions amount to.

Degrees of flexibility in the formal solution are expressed by the presence of integration constants. These are a necessity. Without them the solution could not be subjected to the particular requirements called for by a specific problem. *The number of integration constants must always correspond exactly to the number of initial conditions that we wish to impose.*

This brings us to a very interesting correlation between degrees of freedom of the system, and the maximum number of initial conditions, and integration constants. We have seen that each mesh in an electrical network corresponds to a dimension or coördinate in the mechanical sense, and hence represents a degree of freedom. We shall see later that, for the physical linear network, each additional mesh or degree of freedom raises the order of the differential equation representing the dynamic equilibrium condition in terms of one of the mesh-currents, by a maximum of two. In the simple case discussed above, the network was composed of a single mesh. The differential equation was of the second order. We shall see that, for the physical linear network, the maximum order of the differential equation is twice the number of degrees of freedom.

Now the number of integration constants always corresponds to the order of the differential equation. Our simple circuit above led to two integration constants. Suppose we consider that each additional mesh introduced at least one additional condenser and one additional inductance. Then the order of the differential equation would be double the number of meshes, and this would also be the number of integration constants in the solution. Now for each mesh we are able to specify one initial charge and one initial current, i.e., we can specify a maximum of twice as many initial conditions as there are meshes. But this just corresponds to the number of integration constants. In the case of our simple mechanical analogy the solution contains two integration constants. This just gives it the required flexibility, because at the initial instant or the instant at which the force engages the ball, the latter may have any particular position and velocity. These are the two arbitrary initial conditions. In our analogous electrical case for which eq. (46) is the general solution, we can specify that at the instant the switch S of Fig. 7 introduces the generator, the charge in the condenser and the current flowing in the inductance have any values at all. These are the initial conditions for this simple electrical circuit. Their number just corresponds to the number of integration constants in the solution (46).

This agreement between the number of initial conditions and the number of integration constants is universal. We spoke before of the necessity of having just sufficient integration constants in our solution. The integration constants give the general solution its necessary flexibility provided the degree of flexibility, i.e., the number of integration constants, is just right. If there were a greater or less number of integration constants than available initial conditions — just one more or one less — then it would be impossible to arrive at a unique solution. A solution which is not unique would mean that the network would not always behave in the same manner even though it were always excited under the same conditions. This would be nonsense from physical considerations; or rather it would be a deplorable situation. We could not be sure for instance that an electrically driven street car would always move in the same direction with a given speed for a given controller setting and specified mechanical conditions. This would mean a total collapse of the possibility of utilizing this means of propulsion. Similar situations would be met with in other fields. But fortunately nature does not behave that way. Solutions for physical phenomena are unique, and it is a harmonious characteristic of the science of mathematics that it always concurs with physical laws in providing solutions with just exactly the right number of integration constants for the production of a unique solution. This characteristic has often been pointed out as the inherent harmony between mathematics and physics. When we consider, however, that mathematics is nothing more than a physics in symbolic form, this harmony or intimate relationship no longer seems either accidental or phenomenal. It nevertheless is a wonderful example of the far-reaching completeness with which these two sciences are associated.

8. The Occurrence of Coincident Modes. There remains to discuss one more important point which incidentally further exemplifies this agreement between the mathematical and physical side of the network problem. We shall use our simple R, L, C, circuit of Fig. 7 again. It will be remembered that the determinantal equation for this case was given by:

$$p^2 + \frac{R}{L}p + \frac{1}{LC} = 0.$$

In the preceding paragraphs it was tacitly assumed that the roots of this equation were different from each other, i.e., $p_1 \neq p_2$. This, of course, is the general case. But it may occur that $p_1 = p_2$, that is, the determinantal equation above may be in the form of a full square. Suppose this is the case. Then our transient solution would no longer consist of two terms because these could be collected into one, thus:

$$\begin{aligned} i &= A_1e^{pt} + A_2e^{pt} \\ &= (A_1 + A_2)e^{pt} = Ae^{pt} \end{aligned} \tag{47}$$

where p is the common root $p_1 = p_2 = p$. Our complete solution now consists of the steady-state term as before, plus the transient solution (47). But how do we stand on the question of integration constants? We have apparently only one when we should have two, because we still have two arbitrary initial conditions to specify. It looks as though the harmony between mathematics and physics has suddenly ceased. But this is not the case. It may be remembered that throughout the foregoing discussion nothing was said about any restrictions on the amplitude A. We did not even specify that it must necessarily be a constant. The only real restriction that we impose upon this amplitude A is that it shall be so constituted that the complete solution will be capable of meeting the required conditions of the specific problem treated.

For the particular case where the determinantal equation has equal roots, the amplitude A turns out to be a function of time. This may be shown in a rigorous mathematical fashion. We will not go through this here because we are not interested in rigorous mathematics at the present time. It will suffice to go into this matter in a qualitative way only. In this way we can meet the situation in quite a satisfactory manner.

If A is a function of time, it may be represented by a Taylor series, just as any function of time may be expanded into such a series about a point for which the function is regular and continuous. Thus we would have:

$$A = A_1 + A_2t + A_3t^2 + \cdots . \tag{48}$$

For our purpose we need consider only as many terms as necessary in order to give A a value that will make the resulting complete solution in which it is used capable of meeting the required number of initial conditions. In our present case we need two arbitrary constants of integration. Hence we will retain the first two terms of (48) and discard the rest, i.e., we will let:

$$A = A_1 + A_2t. \tag{49}$$

Thus our complete solution for the case of coincident roots becomes instead of (46):

$$i = \frac{Ee^{j\omega t}}{R + j\left(L\omega - \dfrac{1}{C\omega}\right)} + (A_1 + A_2t)e^{pt}. \tag{50}$$

If we had used a rigorous mathematical method in place of the above qualitative reasoning, we would have arrived at the following condition equation for A:

$$\frac{d^2A}{dt^2} = 0 \tag{51}$$

from which:

$$\frac{dA}{dt} = \text{constant} = A_2 \tag{52}$$

and:

$$\begin{aligned} A &= A_2t + \text{constant} \\ &= A_1 + A_2t \end{aligned}$$

which agrees with (49). In fact it can be rigorously demonstrated that if n of the roots of the determinantal equation coincide, then:

$$\frac{d^nA}{dt^n} = 0 \tag{53}$$

where:

$$Ae^{pt}$$

represents the transient term for these coincident roots. This gives:

$$A = A_1 + A_2t + \cdot \cdot \cdot + A_nt^{n-1} \tag{54}$$

so that the transient term becomes:

$$(A_1 + A_2t + \cdot \cdot \cdot + A_nt^{n-1})e^{pt}. \tag{55}$$

This term contains n constants of integration which again suffices for the unique satisfaction of initial conditions. Thus our harmonious relationship is re-established.

It may seem to the reader that the heuristic method of attack spoken of in this chapter is nothing more than a guess and patchwork process guided more or less by intuition and reasoning based upon past experience and familiar mechanical analogies. This is exactly what it is. Most students do not seem very well satisfied with such a process of obtaining a solution. They rather prefer a so-called rigorous straight-forward integration process. But if they will take the trouble to look into the latter in connection with some of the matters discussed in this chapter they will find that the operations involved are a mere manipulation of mathematical symbols in accordance with prescribed rules. The entire process is as dry as a desert and as translucent as the proverbial clay. It lacks the spirit of initiative and adventure that attends the heuristic method. The latter is so much more intelligent because it requires so much more insight and understanding of the physical principles involved. It requires also a certain "instinct" or "feeling" which comes only with intimate acquaintance with various physical processes and the usual mathematical functions by which they may be described. In short the heuristic method is an inspirational method, and carries with it all the color and enthusiasm which comes from real mental activity. It forms the basis of the true research spirit.

It is true that the real significance of this sort of mental research is hard for the student to develop. Its mainspring is composed of experience and initiative, both of which are still in the early stages of development. But it is just these very qualities which this present treatment proposes to develop.

The reader should early recognize that the process of solving a given problem is nothing more than that of finding some function which will conform with two major requirements: first, that it shall satisfy the demands of the differential equation expressing the equilibrium of the system involved; and second, that it shall be of sufficient flexibility to meet any further conditions such as initial conditions or any other special requirements of the problem. Any method of arriving at this function is legitimate, and the shorter process is always the preferable one. Furthermore, that process which not only gives the desired function but also affords a more or less complete physical understanding of the mechanism involved, will be the most useful.

PROBLEMS TO CHAPTER I

1-1. A network consists of a resistance, a coil, and a condenser in parallel. How many degrees of freedom does this system possess? How many will it possess if the resistance is removed? If the resistance and the inductance are both removed? How many degrees of freedom would be involved if all three elements were connected in series to form a closed circuit?

1-2. What is the analogous electrical circuit for a mechanical pendulum which is subjected to very small displacements?

1-3. What is the electrical analogy of a balanced mechanical flywheel?

1-4. Set up the dynamic equilibrium conditions for the mechanical and electrical systems of Prob. **1-3** for a constant driving force. Repeat for the force-free conditions.

1-5. A rope is passed over a pulley and two different weights tied to the free ends. The system is acted upon by the force of gravity. Neglect the weight of the rope and the moment of inertia of the pulley as well as all friction. How many degrees of freedom does this system possess? Draw the analogous electrical circuit. Set up the condition of dynamic equilibrium for both the mechanical and electrical systems. Sketch the resultant "motions" as functions of time.

1-6. Repeat Prob. **1-5** taking into account the friction of the system, assuming that the frictional force in the mechanical case is a linear function of the velocity.

1-7. The lighter weight of Prob. **1-5** is tied to the floor by a coil spring of known elastance. How many degrees of freedom does the system now possess? Draw the analogous electrical system. Set up the conditions of dynamic equilibrium for both systems. Sketch the approximate behaviors of the systems after they have been subjected to a shock and then left alone.

1-8. Repeat Prob. **1-7** taking into account the frictional forces, assuming them to be linear functions of the velocity in the mechanical system.

CHAPTER II

THE SINGLE MESH NETWORK WITH CONSTANT EXCITATION

1. The R, L Circuit. The simplest kind of circuit to begin on that serves to illustrate some of the things we have been talking about is given in Fig. 8. It consists of a battery E, a switch S, a lumped resistance R, and lumped inductance L in series. Our problem is to find what happens when the switch is closed. We shall call the instant the switch closes $t = 0$. The condition of dynamic equilibrium after $t = 0$ is expressed by the following differential equation:

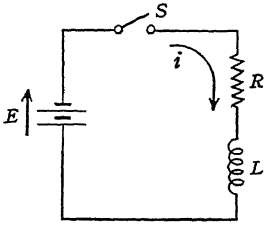


FIG. 8

$$L\frac{di}{dt} + Ri = E. \tag{56}$$

The first term represents the counter e.m.f. of self-induction, the second the counter e.m.f. of resistance, and the right-hand side is the applied constant e.m.f. The equation expresses the fact that the sum of counter e.m.f's. in the circuit equal the impressed force at every instant after the switch closes. Incidentally eq. (56) also holds for the instant $t = 0$.

In the following we shall designate the steady state and transient portions of the complete solution by subscripts. Thus we shall write for the complete current solution:

$$i = i_s + i_t. \tag{57}$$

i_s is the particular integral of (56). Since E is a constant, it is logical to assume:

$$i_s = I \tag{58}$$

where I is a constant. Substitution into (56) gives:

$$0 + RI = E$$

or:

$$I = \frac{E}{R}$$

so that:

$$i_s = \frac{E}{R}. \tag{59}$$

This might have been written down by inspection, but we wished here to show that it was consistent with the differential equation (56).

The transient portion i_t is the solution of the corresponding homogeneous equation:

$$L\frac{di}{dt} + Ri = 0. \tag{60}$$

This may be integrated by separation of variables in the usual fashion, but we prefer to use the direct method. Since an exponential always satisfies this type of equation we write:

$$i_t = Ae^{pt}. \tag{61}$$

Substitution into (60) gives the condition equation:

$$(Lp + R)Ae^{pt} = 0. \tag{62}$$

For the existence of a non-trivial solution we must satisfy the further condition:

$$Lp + R = 0 \tag{63}$$

from which:

$$p = -\frac{R}{L}. \tag{64}$$

Equation (63) is the determinantal equation for this case, and its solution (64) gives the one natural mode of this system. We see that it is purely real. This means that the current variation will be monotonic. Furthermore this natural root (64) is negative. This means that the natural behavior of the system is a decay and will eventually reach a rest condition. This is consistent with physical considerations, because without a driving force the system must eventually come to rest on account of the frictional dissipation of energy. In fact all networks involving positive resistances, inductances, and capacitances will give rise to natural roots or modes with negative real portions for the reason just mentioned.

Our transient solution therefore becomes:

$$i_t = Ae^{-\frac{R}{L}t}. \tag{65}$$

The complete solution is the sum of (59) and (65), thus:

$$i = \frac{E}{R} + Ae^{-\frac{R}{L}t}. \tag{66}$$

This solution contains one integration constant. This is as it should be since we have just one initial condition to specify, namely the current in the circuit at $t = 0$. The circuit contains no condenser, so that there is no initial charge to specify.

Let us say for the time being that the current at the instant $t = 0$ has the value i_0. Substituting this for i in (66) and putting $t = 0$ we have:

$$i_0 = \frac{E}{R} + A$$

from which:

$$A = i_0 - \frac{E}{R} \tag{67}$$

so that the final form for our solution becomes:

$$i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) + i_0 e^{-\frac{R}{L}t}. \tag{68}$$

In our case the switch S is open for $t < 0$. Hence there can be no current flowing before the switch closes. At the instant $t = 0$ the battery starts to push electrical charge through the coil and resistance. This is like a force suddenly applied to a mass which slides upon a surface with friction as illustrated in Fig. 9. The equation of dynamic equilibrium analogous to (56) for this mechanical system is:

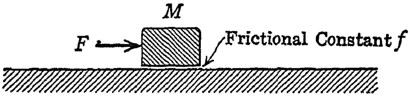


FIG. 9

$$M\frac{dv}{dt} + fv = F$$

where v is the velocity of motion, or the analog of current in the electrical case. Now we know from mechanics that the body M cannot assume a finite velocity instantaneously, assuming that it was at rest before the force F was applied. A discontinuous velocity, i.e., a jump in velocity from zero to a finite value would mean an infinite acceleration at the initial instant, and this in turn would call for an infinite force. Hence if the velocity was zero before the force took hold, it will still be zero immediately after that instant. This is a fundamental fact, namely that in a system containing mass, the velocity of motion must at all times be continuous unless the disturbance is in the nature of an impact. This means that the curve of velocity plotted against time may nowhere have a vertical jump or discontinuity. It may have a kink, i.e., an abrupt change in slope or direction. In fact it will have such an abrupt change in slope wherever the impressed force is discontinuous, as for example, at the initial instant in the case considered above.

Just so, the current in an electrical circuit containing inductance cannot be discontinuous because this would mean an infinite $\frac{di}{dt}$, which in turn would mean an infinite voltage of self-induction. The current curve plotted against time may on the other hand have a sudden change in slope, i.e., a discontinuous first derivative. It will have such an abrupt change in slope wherever the impressed e.m.f. suddenly changes.

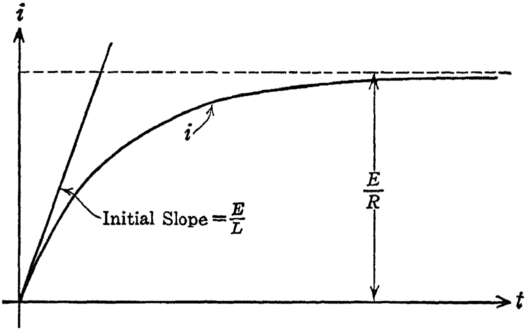


FIG. 10

From this it will be clear that for our problem we have to put $i_0 = 0$ in the final expression (68). However, it may very well be that a circuit will be so arranged that current can flow before the switching instant, in which case the current will be continuous throughout this instant, and hence i_0 will have a value. In that case (68) is the expression to use with the proper value substituted for i_0. The reader may draw the proper circuit arrangement to illustrate this case for himself.

For the solution of the case illustrated in Fig. 8 we therefore have:

$$i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right). \tag{68a}$$

This curve is plotted in Fig. 10. The final steady state is the horizontal line E/R, which is the asymptote of the actual current i. The current never actually reaches this value but practically does so in a very short time. In fact after an elapse of time equivalent to:

$$\frac{Rt}{L} = 1 \tag{69}$$

the current will have reached:

$$\frac{i}{E/R} \cdot 100 = (1 - e^{-1}) \cdot 100 = 63.2\%$$

of its final value. This length of time which is given by:

$$t = T = \frac{L}{R} \tag{69a}$$

is commonly called the **time constant** of the circuit and is usually in the neighborhood of a small fraction of a second for ordinary circuits. For instance if the coil has an inductance of one millihenry and a resistance of one ohm, the time constant T would be 0.001 second. Hence in such a coil the current would reach 63.2% of its final value in one-thousandth of a second.

The function of the transient portion of the solution is obviously to smooth out the initial stages. The steady state alone demands that the current shall instantly jump to the value of E/R. The physical characteristics of the circuit however, due to the inertia effect of the inductance, preclude the possibility of such a jump in the current. The transient steps in to smooth things over and tide the steady state before $t = 0$, which in our case is a rest condition, over into the steady state after $t = 0$. We will find that in the more complicated cases to be taken up later, this is universally true. The transient terms act as shock absorbers. They absorb the discrepancy between what the current is at the switching instant, and what the steady state demands that it should be.

Before leaving this case, it is interesting to see what would happen in the limiting case $R \rightarrow 0$. If we substitute this value into eq. (68), it becomes indeterminate. This situation can be alleviated by an expansion of $e^{-\frac{Rt}{L}}$ in its power series and dropping all powers of R above the second. Thus for very small resistances the following form would be approximately correct:

$$\begin{aligned} i &= \frac{E}{R}\left(1 - 1 + \frac{Rt}{L} - \frac{R^2t^2}{2L^2} + \cdots\right) \\ &= \frac{Et}{L} - \frac{ER}{2L^2}t^2 + \cdots. \end{aligned}$$

For the limit $R \rightarrow 0$ we then have:

$$i = \frac{E}{L}t. \tag{70}$$

This is a straight line through the origin with a slope of $\frac{E}{L}$ amperes per second. Looking back at Fig. 10, we see that the current for the case with resistance present starts out along this straight line but soon bends away from it and runs off asymptotic to the value $\frac{E}{R}$. Now that R is zero however the asymptote is removed to infinity, and the curve does not bend but continues straight on up toward infinity. It must be remembered however that such a case cannot be realized physically, and that we are giving it here only for the sake of interest and the additional light it may throw upon the actual physical case originally discussed.

2. The R, C Circuit. We now turn our attention to the circuit given in Fig. 11. This consists of an e.m.f. E, a switch S, a resistance R, and a condenser C in series. The equation expressing the condition of equilibrium between the impressed and the counter-voltages after S closes now takes the following form:

$$Ri + \frac{1}{C}\int i\,dt = E. \tag{71}$$

The novel feature about this equation is the appearance of the integral representing the accumulated charge in the condenser at any time. We do not have to solve an equation in order to recognize that the steady-state current in this case is zero. However, if we wish to see what our mathematical methods have to say about the matter, let us suppose that the steady-state current is any constant, thus:

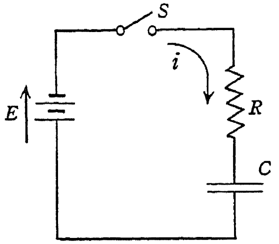


FIG. 11

$$i_s = I. \tag{72}$$

Substituting into (71) we get:

$$RI + \frac{I}{C}t + \text{const.} = E. \tag{73}$$

Every term in this equation is a constant except the second, which varies linearly with time. It is not hard to see that the only value of I which will satisfy eq. (73) for all values of time is:

$$I = 0.$$

We expected this of course since no steady current can flow through a condenser.

Since there is no steady state, the transient portion will be the whole solution. For this we must solve the corresponding homogeneous equation to (71), namely:

$$Ri + \frac{1}{C}\int i\,dt = 0. \tag{74}$$

As usual for all circuit problems we expect that:

$$i_t = Ae^{pt} \tag{75}$$

will be a solution. Substituting this value into (74) we get the condition equation:

$$\left(R + \frac{1}{Cp}\right)Ae^{pt} = 0 \tag{76}$$

from which the existence of a non-trivial solution demands that:

$$R + \frac{1}{Cp} = 0. \tag{76a}$$

This is the determinantal equation for this case, from which the natural mode is found by solving for p, thus:

$$p = -\frac{1}{RC}. \tag{77}$$

The natural mode is purely real and negative, just as in the preceding case. The transient solution — or complete solution in this case — therefore becomes:

$$i_t = Ae^{-\frac{t}{RC}}. \tag{78}$$

Again we have only one integration constant. This is as it should be because there is only one initial condition to specify, namely the charge in the condenser at the time the switch closes which we again call $t = 0$.

The initial current cannot be specified without some analysis or reasoning because the circuit has no inductance. This is like suddenly applying a constant force to a spring having internal friction but no mass. There is no inertia effect, and hence the velocity, or its electrical analog the current, can be discontinuous at the instant the force is applied. In fact it must be in this case. The applied force must be balanced by an equal and opposite force at the initial instant as well as at any other, and since the condenser cannot offer this opposing force unless it happens to have just the right charge, the required counter-e.m.f. must instantly appear across the resistance. This, of course, requires that the current jump instantly to the proper value. Incidentally it is quite impossible for the charge in the condenser to change instantly because this would require an infinite current at the initial instant, which, of course, would cause an infinite Ri drop. This is due to the fact that charge is the integral of current, and in order to accumulate a finite charge in no time, an infinite rate of flow is required.

In order to evaluate the integration constant in (78), let us suppose that the condenser has a charge of q_0 coulombs in it before the switch closes. However, we cannot use this fact directly for the evaluation of the integration constant in eq. (78) because the latter gives the current as a function of time and not the charge. In order to determine what the initial current is we must go back to the original differential equation (71). If we consider this at the time $t = 0$, then the value of the integral becomes the initial charge. Hence we have:

$$R(i)_{t=0} + \frac{q_0}{C} = E \tag{79}$$

from which:

$$(i)_{t=0} = \frac{E}{R} - \frac{q_0}{RC}. \tag{80}$$

This information can now be used directly in eq. (78) to evaluate the constant A. It may be said in passing that whenever a difficulty arises that cannot be answered by inspection, the proper thing to do is to go back to the original differential equation. This contains all the information except initial conditions, and will usually be in a position to give the required answer. In the present case for instance we needed the initial current, but had the initial charge instead. By going back to eq. (71) we obtained the answer quite easily.

Writing (78) for $t = 0$, we then have by (80):

$$\frac{E}{R} - \frac{q_0}{RC} = A. \tag{81}$$

Thus the final solution to our problem becomes:

$$i = \left(\frac{E}{R} - \frac{q_0}{RC}\right)e^{-\frac{t}{RC}}. \tag{82}$$

We see that this solution will vanish for:

$$q_0 = EC \tag{83}$$

which is the charge necessary to bring the condenser up to a voltage equal to that which is suddenly impressed. In Fig. 12 we show a plot of eq. (82) for a case where the initial charge is zero. The student may illustrate other cases for himself.

The solution consists merely of a sudden rise to the value E/R, followed by an exponential decay. Circuits of this kind, where the current is allowed to jump instantaneously, are called **impulsive.**

Sometimes the problem discussed here is varied somewhat regarding the method of excitation. That is, the condenser is assumed to be initially charged. There is no impressed e.m.f.

The circuit is merely closed upon itself, thus discharging the condenser through the resistance. We have however already solved this discharge case, as well as an infinity of intermediate cases. Equation (82) gives the solution for any impressed voltage and initial charge. If we wish to have the discharge case, we need merely put $E = 0$ and give q_0 the required initial value.

It should be noted that the function of the transient in this case is also in the nature of a shock absorber. The steady-state condition is not adequately represented by the equation $i = 0$. It is however completely specified by $q = EC$ where q is the charge in the condenser. The latter has a constant value in the steady state. The transient appears only when a discrepancy exists between this steady-state value and that which is actually present at the instant the switch closes. Again the action of the transient is to smoothly carry the system over from the existing condition to that demanded by the steady state.

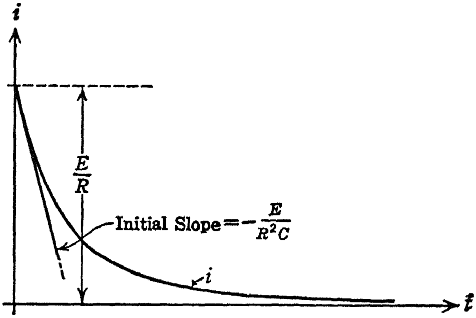


FIG. 12

The usual definition for the time constant, as explained in the previous case, applies to this circuit. We have for this case: $T = RC$ seconds.

Before proceeding with the next case, let us summarize the outstanding features of the two simple cases discussed so far. The reader will probably appreciate already that the hardest part about determining network solutions is the evaluation of integration constants in agreement with specified initial conditions. In order to save time in this process it is well to develop certain useful rules. Take the inductive circuit of Fig. 8 for example.

For this case we saw that the initial current was zero. This was due to the presence of the inductance. Hence we can say that an inductance acts like an open circuit for the first instant. In the capacitive case of Fig. 11, on the other hand, the initial current for zero initial charge was E/R. Hence we say that a condenser acts like a short circuit for the first instant. It should be further remembered that these rules hold only for the case where the condition before the switching operation is a rest condition, by which we mean that no current flows and no charges are present before the switching operation takes place. If charges are present, or if the circuit is so arranged that current may be flowing before the switching operation considered, then these rules are, of course, invalid unless special provisions are made to take care of the existing charge or current. The usual conditions, however, are rest conditions, so that the above rules very frequently may be applied in order to save time.

3. The L, C Circuit. The next case in order is that shown in Fig. 13. This consists of a series arrangement of an e.m.f. E, a switch S, an inductance L, and a capacitance C. The equation which expresses the condition of dynamic equilibrium for this case is:

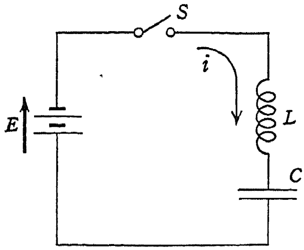


FIG. 13

$$L\frac{di}{dt} + \frac{1}{C}\int i\,dt = E. \tag{84}$$

We can see immediately that the steady-state current will be zero just as it was in the case of the R, C circuit. Hence the transient portion is the complete solution again. This is the solution of the force-free equation:

$$L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0. \tag{85}$$

It may seem queer that the actual equation of equilibrium with the force impressed is apparently not used to find the solution. This is not the case, however. We shall have to come back to eq. (84) in order to determine the integration constants. Thus the e.m.f. E will eventually have its effect upon the solution although it seems for the time being that it is being disregarded.

For the solution of (85) we make the usual assumption:

$$i = Ae^{pt}. \tag{86}$$

Substitution gives the condition equation:

$$\left(Lp + \frac{1}{Cp}\right)Ae^{pt} = 0. \tag{87}$$

Demanding a non-trivial solution requires that:

$$Lp + \frac{1}{Cp} = 0 \tag{88}$$

which is the determinantal equation for this case. It is equivalent to the following quadratic:

$$p^2 + \frac{1}{LC} = 0 \tag{88a}$$

which has the roots:

$$\left.\begin{aligned} p &= \frac{\pm j}{\sqrt{LC}} = \pm jg_0 \\ g_0 &= \frac{1}{\sqrt{LC}}. \end{aligned}\right\} \tag{89}$$

This is the first case for which we have obtained imaginary roots or natural modes. We recognize therefore that the solution is going to be oscillatory instead of monotonic as the previous cases were. The quantity g_0 is called the **undamped angular frequency.** The reason for calling it *undamped* will become clear after we treat the same case with resistance considered.

Since the determinantal equation is a quadratic, we have a pair of natural modes, differing however merely by an algebraic sign, as shown by eq. (89). Each of these roots will give rise to a solution of the form of (86). Hence the sum of two such terms represents the complete transient solution, and therefore the complete solution for this case. We have:

$$i = A_1e^{jg_0t} + A_2e^{-jg_0t}. \tag{90}$$

This time we have two integration constants, A_1 and A_2. We have however two initial conditions to specify, so that two integration constants are necessary in order to provide a unique solution.

We will assume initial rest conditions this time, i.e., that the charge in the condenser and the current in the inductance are zero for $t = 0$. The latter must, of course, be true since the switch is open initially. The condenser could have a charge in it to start with, but we shall leave this case for the student to work out as an exercise. Thus our initial conditions become:

$$\left.\begin{aligned} (i)_{t=0} &= 0 \\ \text{and:} \quad \left[\int i\,dt\right]_{t=0} &= 0. \end{aligned}\right\} \tag{91}$$

The first of these is directly applicable to eq. (90). The second is not. In order to use this we must go back to the original equation (84) and consider it for the instant $t = 0$. This gives:

$$\left(\frac{di}{dt}\right)_{t=0} = \frac{E}{L} \tag{92}$$

which is applicable to (90) after we differentiate once with respect to time and then set $t = 0$. The result (92) might also have been obtained from the rule that the condenser acts like a short circuit for the first instant.

Using the first eq. (91) and eq. (92) in connection with (90), we obtain:

$$\begin{aligned} A_1 + A_2 &= 0 \\ A_1 - A_2 &= \frac{E}{jLg_0}\cdot \end{aligned} \tag{93}$$

The solution of this pair of equations gives:

$$\left.\begin{aligned} A_1 &= \frac{E}{2\,jLg_0} \\ \text{and:} \quad A_2 &= -\frac{E}{2\,jLg_0}\cdot \end{aligned}\right\} \tag{94}$$

Substituting these values back into (90), we have:

$$i = \frac{E}{Lg_0}\cdot\frac{e^{jg_0t} - e^{-jg_0t}}{2j}\cdot$$

But:

$$\frac{e^{jx} - e^{-jx}}{2j} = \sin x$$

so that:

$$i = \frac{E}{Lg_0} \sin g_0 t. \tag{95}$$

There are several important things to note before proceeding with a graphical discussion of this result. First it should be recognized that, whereas the determinantal equation gave rise to two natural modes as given by eq. (89), the final result (95) contains only one frequency. This is due to the fact that the roots (89) are a conjugate pair. This is fundamental in network theory. When imaginary or complex roots occur, they always come in conjugate pairs, and then a pair of roots defines only one real frequency, just as in this simple case. Second it should be emphasized that the pair of integration constants that go with the terms involving a pair of conjugate roots, are themselves a pair of conjugates, as illustrated by eq. (94). This is also universally true of network theory. The reason for this second point may be seen from eq. (90). The current on the left is certainly real. This means that the sum of the two right-hand terms must be real. But this can be so only when the two terms are a pair of conjugates. Since the exponentials are conjugates on account of the conjugate exponents, it follows that A_1 and A_2 must be conjugates.

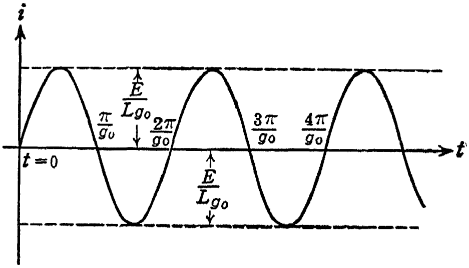


FIG. 14

Let us now discuss the solution (95). This result is plotted in Fig. 14. It is rather striking in that the current starts right in and oscillates with a constant amplitude at the angular frequency of g_0 radians per second. There is no introductory period. It certainly illustrates quite forcibly how eager an L, C circuit is to oscillate at its natural frequency. It would hardly be supposed that a unidirectional force could so easily be converted into any desired harmonic oscillation. The fact that it can, however, is quite fundamental in communication work, especially in radio work. This principle forms the basis of the well-known spark oscillator of the early days of radio, as well as that of many later developments for the production of high frequency oscillations by means of shock excitation. It must be remembered, of course, that we have not yet considered the effect of resistance upon the resulting oscillation. We shall see that this effect is in the nature of a damping action such that the oscillations persist only for a few cycles unless the application of force is repeated. This will be amply illustrated later on.

Incidentally if an actual physical circuit could be so influenced as to neutralize its resistance, then it would continue to oscillate with undiminished amplitude. This fact forms the basis of such oscillation generators as the arc and some vacuum tube circuits, which are essentially resistance neutralizing devices. It is, of course, not the object of this text to present these matters, but the important bearing which the theory of the simple L, C circuit has upon such matters should come to the student's attention at this time.

It is further illuminating to discuss the voltages appearing across the inductance and the capacitance as functions of time. The analytical expressions for these functions are easily obtained. The voltage induced in the inductance is:

$$e_i = L\frac{di}{dt} \tag{96}$$

and that appearing across the condenser is:

$$e_c = \frac{1}{C}\int i\,dt. \tag{97}$$

Substituting (95) into (96) and (97), we get:

$$e_i = E\cos g_0 t \tag{98}$$

and:

$$e_c = -E\cos g_0 t + \text{const.} \tag{99}$$

The constant of integration appearing in the last equation is easily evaluated, if we consider that $e_c = 0$, for $t = 0$. Thus we get:

$$e_c = E(1 - \cos g_0 t). \tag{99a}$$

Fig. 15 shows plots of e_i and e_c against time. This figure is interesting for several reasons. The inductance voltage oscillates about zero with an amplitude equal to the battery voltage. The condenser voltage on the other hand oscillates about the value E with an amplitude equal to E, so that it goes from zero to twice E and back to zero. Note that the sum of e_i and e_c is equal to the constant battery voltage E at every instant, as of course it should be. This is required by the dynamic equilibrium of forces of which our original differential equation (84) was an expression.

The action of the condenser voltage has a physical meaning that is worth noting. If the oscillation process could ever settle down some way, it would stop with the condenser charged to the battery voltage. In fact this is the object which the battery is trying to accomplish, but it never gets there because the condenser voltage overshoots its mark by the amount E, then decreases and again overshoots its mark by the same amount, etc. The process continues forever due to the fact that there is no stabilizing influence present in the way of frictional forces.

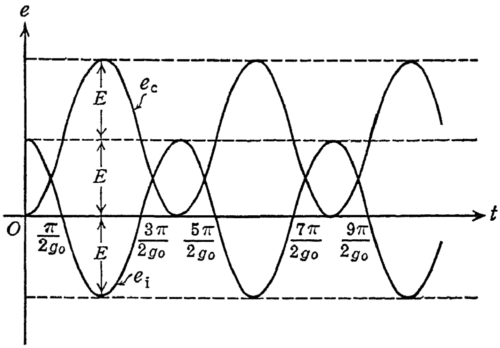


FIG. 15

4. Vector Treatment for the L, C Circuit. There is another much more powerful way of treating this simple case that lends itself to a more varied class of applications. The relative importance of the simple oscillatory circuit justifies this additional discussion. This method is based upon a very convenient way of representing a harmonic function. Harmonic motion results when the extremity of a vector of constant amplitude, rotating with constant angular velocity, is projected upon a fixed axis.

The function $e^{j\omega t}$ represents such a vector of unit length rotating with an angular velocity of ω radians per second in the Gaussian plane, commonly called the complex plane or the plane of complex numbers. This is illustrated in Fig. 16. Here we also show that the vertical projection is a sine function while the horizontal projection is a cosine function. This is expressed by the trigonometrical relation:

$$e^{j\omega t} = \cos \omega t + j \sin \omega t.$$

Thus if we wish to represent a sine function analytically, we may do so by recognizing that:

$$\mathcal{I}_m[e^{j\omega t}] = \sin \omega t.$$

Similarly the cosine function would be:

$$\mathcal{R}_e[e^{j\omega t}] = \cos \omega t.$$

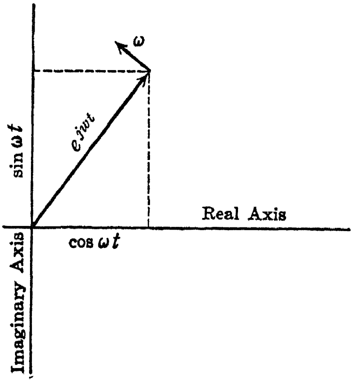


FIG. 16

The symbols $\mathcal{I}_m$ and $\mathcal{R}_e$ are operators which perform the operation of taking the imaginary and real portions respectively of what they operate upon.

If we wish to represent a sine function, for instance, of any amplitude, we merely write:

$$\mathcal{I}_m[Be^{j\omega t}] \tag{100}$$

where B is the desired amplitude. An additional point of exceptional value, and one which is very little used in engineering literature, is the fact that eq. (100) not only represents a sine function of ωt having any amplitude, but having any relative phase angle with respect to some reference as well. This becomes a fact, if we consider B to be a complex number, i.e., a complex amplitude. If this is so, then we can write:

$$\left.\begin{aligned} B = B_1 + jB_2 &= |B| \underline{|\phi} = |B|e^{j\phi} \\ \text{where:} \qquad \tan \phi &= \frac{B_2}{B_1}. \end{aligned}\right\} \tag{101}$$

Thus (100) really is:

$$\mathcal{I}_m[|B|e^{j(\omega t+\phi)}] = |B| \sin (\omega t + \phi). \tag{102}$$

This is an extremely useful artifice that the student will appreciate only after considerable familiarity with its use. Phase angles need never be written out in the usual cumbersome fashion. They are contained in the form (100) without appearing explicitly. Hence this form saves a lot of confusion and bother in handling problems involving arbitrary phase angles.

To get back to our L, C circuit problem, we see that the current given by eq. (95) could be written as:

$$i = \mathcal{I}_m[Be^{jg_0t}]. \tag{103}$$

In fact eq. (103) could represent the oscillation in any phase position by merely supplying B with the proper angle. For instance suppose B has an angle of $\frac{3\pi}{4}$ radians and a magnitude equal to

$$\frac{E}{Lg_0}.$$

Then (103) represents (95) advanced in phase by 135° as illustrated in Fig. 17. Here the imaginary axis is considered as the P. A. or projection axis. This is a most convenient way of representing an oscillatory function of arbitrary amplitude and phase.

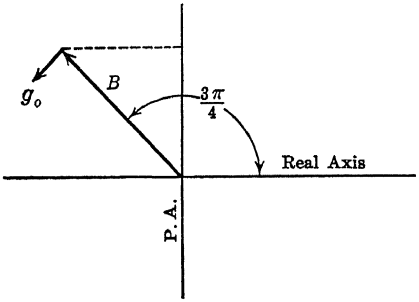


FIG. 17

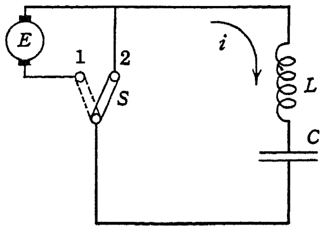


FIG. 18

We are now ready to apply the heuristic method to the oscillatory L, C circuit in a very general way. Instead of Fig. 13 we now consider the circuit arrangement given in Fig. 18. This differs from Fig. 13 merely in that a different switching arrangement is provided. When the switch S is in the position 2, the circuit is closed upon itself, while with S in position 1, the voltage source E is inserted. We shall assume that the switch S may be changed from position 1 to 2 or vice versa instantaneously.

Our problem is now this: Suppose S is in position 2, and that the circuit is oscillating. At some time, which we designate as $t = 0$, the switch is changed to position 1. What is the new condition of oscillation in terms of the old? The heuristic method together with the vector method of representing an oscillating function will give the answer to this question very easily.

It is clear from the solution (95) that the reaction of an L, C circuit to the application of a constant force is to oscillate at its natural frequency with an amplitude proportional to the applied force. If we subject an oscillating circuit to a sudden change of constant applied force, it is logical to expect that after the shock the circuit will still oscillate harmonically at the same frequency. An L, C circuit can do nothing else under the influence of constant forces. The only thing that the sudden change in applied force can do is to change the amplitude of oscillation in magnitude and phase. Suppose therefore that the circuit of Fig. 18 is oscillating with the switch in position 2, and that the expression for current is given by:

$$i_b = \mathcal{I}_m[Be^{j\omega_0 t}]. \tag{103a}$$

At the instant $t = 0$ the switch is snapped to position 1. After $t = 0$ the current may be represented by:

$$i_a = \mathcal{I}_m[Ae^{j\omega_0 t}]. \tag{104}$$

Our problem is to determine A in terms of B. This we do by applying certain conditions which must hold at the switching instant. Since the circuit is inductive, we know that the current cannot be discontinuous, i.e., we know that:

$$(i_a)_{t=0} = (i_b)_{t=0}. \tag{105}$$

At the switching instant the impressed force suddenly changes. Since we do not wish to restrict this discussion in any way, let us denote the change in the impressed force by:

$$\Delta E$$

where this change may be in either the positive or negative direction.

If the impressed force suddenly changes at $t = 0$, then the sum of counter-e.m.f's. in the circuit must also suddenly change at the same time. This sudden change cannot take place in the condenser because this would mean a discontinuity in the charge. Hence it must appear across the inductance. This is permissible because a discontinuous induced voltage in L merely requires a discontinuity in the $\frac{di}{dt}$, and does not violate the condition (105). Hence we see that:

$$L\left(\frac{di_a}{dt}\right)_{t=0} - L\left(\frac{di_b}{dt}\right)_{t=0} = \Delta E. \tag{106}$$

The conditions (105) and (106) just suffice to determine A in terms of B. We need two conditions because these amplitudes are complex, and hence contain two parts, thus:

$$\left.\begin{aligned} A &= A_1 + jA_2 \\ B &= B_1 + jB_2. \end{aligned}\right\} \tag{107}$$

Applying (105) to (103a) and (104), we see that:

$$\mathscr{I}_m[A] = \mathscr{I}_m[B]$$

or:

$$A_2 = B_2. \tag{108}$$

Applying (106), we get:

$$\mathscr{I}_m[Ajg_0] - \mathscr{I}_m[Bjg_0] = \frac{\Delta E}{L}$$

from which:

$$A_1 = B_1 + \frac{\Delta E}{Lg_0}. \tag{109}$$

Multiplying (108) by j and adding it to (109), we finally get:

$$A = B + \frac{\Delta E}{Lg_0}. \tag{110}$$

Thus we see that the amplitude after the disturbance equals the amplitude before the disturbance plus a real quantity which depends upon the magnitude and direction of the applied force.

This result is illustrated in Fig. 19 which shows the vector representation at the left and the corresponding current vs. time picture at the right. The origin is arbitrarily taken to correspond to the vector B in the reference position. The instant at which the applied force changes is designated as $t = 0$. The direction of the force is taken as positive, i.e., a positive e.m.f. is suddenly applied at $t = 0$. We note several interesting things. First, the amplitude is decreased in magnitude by the application of an e.m.f. Second, the phase is suddenly set back by the amount φ. This result is due to two things: First, the choice of the switching instant; and second, the direction or algebraic sign of E. Obviously, if the switching instant had been chosen earlier, when B was in the first quadrant, the amplitude A would have been larger than B. If on the other hand an e.m.f. had been suddenly removed instead of inserted, then A would also have been larger than B, and its phase would have been advanced instead of retarded.

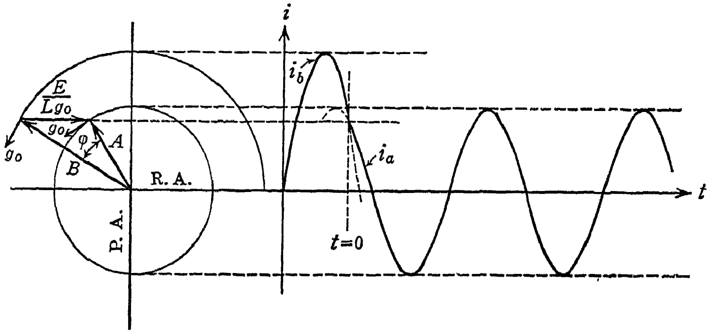


FIG. 19

The reader will agree that this method of representation immediately gives us a complete survey of all possible cases that might arise in connection with the problem of the sudden application or removal of constant forces from an L, C circuit in the process of oscillation. By properly chosen switching instants the oscillation may either be built up or wiped out, and the phase may be advanced or retarded by any amount up to π radians.

A physical picture of this may be had, if we consider a mechanical pendulum. By judiciously applied and removed pushes the motion of the pendulum may be built up or down. We will let the reader exhaust the various possibilities of these ideas as practice problems.

5. The R, L, C Circuit. We proceed with the next problem in order, which is the same as that just treated except that we shall now consider the effect of ohmic resistance. This situation is illustrated in Fig. 20. We shall assume that the switch S is thrown from position 2 to 1 at $t = 0$. The positive direction of E coincides with the assumed positive direction of i. For $t \geqq 0$ the following differential equation must hold:

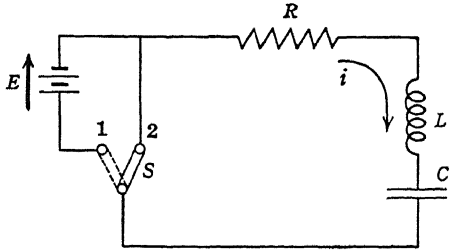


FIG. 20

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E. \tag{111}$$

The particular integral or steady state is obviously again zero. The transient is therefore the complete solution. The only effect that E has upon the solution is therefore to influence the amplitude of oscillation by affecting the evaluation of integration constants.

We therefore proceed to find the transient solution, i.e., the solution of the corresponding force-free equation:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0. \tag{112}$$

The solution must be of the form:

$$i = Ae^{pt}. \tag{113}$$

Substitution into (112) results in the condition equation:

$$\left(Lp + R + \frac{1}{Cp}\right)Ae^{pt} = 0 \tag{114}$$

and the existence of non-trivial solutions demands:

$$Lp + R + \frac{1}{Cp} = 0$$

which is equivalent to:

$$p^2 + \frac{R}{L}p + \frac{1}{LC} = 0. \tag{115}$$

This is the determinantal equation for this case. We recognize:

$$\frac{1}{LC} = g_0^2$$

as the square of the undamped angular frequency which appeared in the previous case. The resistance causes the appearance of a linear term in p. Let us introduce the factor:

$$\alpha = \frac{R}{2L} \tag{116}$$

for reasons which will soon become clear. Then (115) may be written:

$$p^2 + 2\,\alpha p + g_0^2 = 0. \tag{115a}$$

The solution of this quadratic will give us the natural modes of oscillation and decay of the circuit. Using the quadratic formula, we get:

$$p = -\alpha \pm \sqrt{\alpha^2 - g_0^2}$$

or:

$$\left.\begin{aligned} p_1 &= -\alpha + \sqrt{\alpha^2 - g_0^2} \\ p_2 &= -\alpha - \sqrt{\alpha^2 - g_0^2}. \end{aligned}\right\} \tag{117}$$

We cannot proceed with the solution in the usual fashion without introducing some remarks about these roots (117). The square root involves the difference of two positive numbers. If the first is larger than the second, then the square root is real. If the two are equal, then the square root vanishes. If the second is larger than the first, then the number under the radical is negative, and the square root is a pure imaginary. The first of these conditions means two real values for p. The second means that the two values of p coincide in the real value $-\alpha$. The third means that the p's are a pair of conjugate complexes. Now we know that if the natural modes p_1 and p_2 are real, then the resulting current solution will be non-oscillatory, while if the p's are a pair of conjugate complexes, the solution will be oscillatory. This is so because e raised to a real power times t does not represent a harmonic function. The p's must either be pure imaginaries or else have imaginary portions in order to provide a factor e raised to an imaginary power times t. The latter, of course, represents harmonic functions and hence an oscillatory response.

Therefore it is clear that we must separate our result into three cases. Let us consider first the **case I:**

$$\alpha > g_0.$$

Then p_1 and p_2 are both negative reals. Our solution therefore becomes:

$$i = A_1 e^{(-\alpha+\beta)t} + A_2 e^{(-\alpha-\beta)t} \tag{118}$$

where

$$\beta = \sqrt{\alpha^2 - g_0{}^2}$$

is used as an abbreviation.

At $t = 0$ we have two initial conditions to specify, the charge in the condenser and the current in the inductance. For the sake of simplicity let us assume that these are both zero. The method of evaluating integration constants is the same whether we make this special assumption or not. The condition of zero charge at $t = 0$ cannot be utilized in (118) directly. Consequently we go back to the original differential equation and consider it for $t = 0$. At this instant the current term and the integral term vanish. If the initial current and charge had not been assumed zero, then these terms would have known constant values for $t = 0$ instead of being zero. Thus we get:

$$\left(\frac{di}{dt}\right)_{t=0} = \frac{E}{L}. \tag{119}$$

This together with:

$$(i)_{t=0} = 0 \tag{119a}$$

may be used directly in (118) to evaluate A_1 and A_2. We get:

$$\left.\begin{aligned} 0 &= A_1 + A_2 \\ \frac{E}{L} &= A_1(-\alpha + \beta) + A_2(-\alpha - \beta). \end{aligned}\right\} \tag{120}$$

The solution of these equations gives:

$$\left.\begin{aligned} A_1 &= \frac{E}{2\,L\beta} \\ A_2 &= \frac{-E}{2\,L\beta} \end{aligned}\right\} \tag{121}$$

so that the solution (118) finally becomes:

$$i = \frac{Ee^{-\alpha t}}{L\beta}\sinh \beta t \tag{122}$$

because:

$$\frac{e^{\beta t} - e^{-\beta t}}{2} = \sinh \beta t.$$

The plot of (122) is shown in Fig. 21. It consists merely of a pulse of current sufficient to charge the condenser up to the voltage E. After that a condition of static equilibrium exists. If the condenser had had an initial charge $Q_0 = EC$, then there would have been no surge at all. The student should work out this case and convince himself that the solution vanishes for an initial charge of this value.

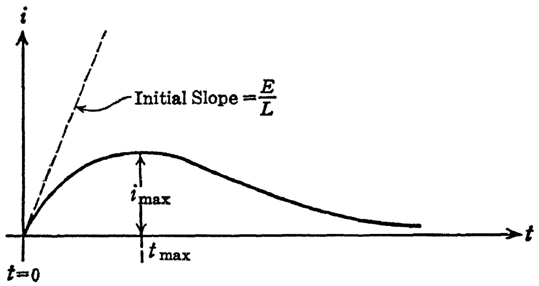


FIG. 21

It is interesting to work out the time at which the maximum current occurs, as well as the value of the maximum. For this purpose we form from (122):

$$\frac{di}{dt} = \frac{Ee^{-\alpha t}}{L\beta}(\beta \cosh \beta t - \alpha \sinh \beta t) = 0$$

from which:

$$t_{\text{max}} = \frac{1}{\beta}\tanh^{-1}\left(\frac{\beta}{\alpha}\right) \tag{123}$$

and:

$$i_{\text{max}} = \frac{E}{Lg_0}\left(\frac{\alpha + \beta}{g_0}\right)^{-\frac{\alpha}{\beta}}. \tag{124}$$

We consider next the **case II**:

$$\alpha = g_0.$$

For this case $\beta = 0$, and $p_1 = p_2 = -\alpha$. We can obtain the solution for this case directly from the preceding by carrying the limit $\beta \to 0$. In eq. (122) we note that for small values of β:

$$\sinh \beta t \to \beta t$$

so that in the limit we get:

$$i = \frac{E}{L} t e^{-\alpha t} \tag{125}$$

which is the solution for this case. A plot of this result is shown in Fig. 22. It looks very much like the curve of Fig. 21 for the preceding case. The initial slope is the same. This must be so because the initial slope times the inductance must equal the impressed voltage. The maximum now occurs for:

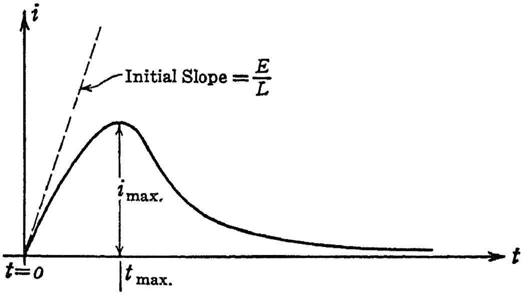


FIG. 22

$$\frac{d}{dt}\frac{E}{L} \cdot t \cdot e^{-\alpha t} = \frac{E}{L}(-\alpha t e^{-\alpha t} + e^{-\alpha t}) = 0$$

or:

$$t_{\max} = \frac{1}{\alpha}. \tag{126}$$

The maximum current is given by:

$$i_{\max} = \frac{Ee^{-1}}{2R} = 0.184\frac{E}{R}. \tag{127}$$

The essential difference between this case and the preceding is that in the present case the current drops back to zero faster than in the former. This can be recognized by comparing the equations (122) and (125) for the two cases. Both consist of the product of two functions of time — an exponential decreasing function, and an increasing function which is a hyperbolic sine on the one hand and simply t on the other. Since the hyperbolic sine increases faster with time than t alone, the whole function (125) will drop off sooner and faster than the function (122). In the present case then the surge is over in less time.

Consider now **case III**:

$$\alpha < g_0.$$

This is by far the most important, because it is the oscillatory case. For this one the natural modes become:

$$p = -\alpha \pm j\sqrt{g_0^2 - \alpha^2}.$$

Here let us introduce the notation:

so that:

$$\left.\begin{aligned} g &= \sqrt{g_0^2 - \alpha^2} \\ p_1 &= -\alpha + jg \\ p_2 &= -\alpha - jg. \end{aligned}\right\} \qquad (128)$$

We call g the **damped angular frequency** in contrast to g_0 which is the undamped angular frequency. It is clear from the first equation (128) that g is always less than g_0. In other words, the resistance effectively lowers the frequency at which the circuit oscillates. Usually α is small, compared to g_0, so that g is not much smaller than g_0. Such circuits for which $\alpha \ll g$, are called **highly oscillatory.** Their behavior is very nearly like that of the L, C circuit previously discussed.

For this oscillatory case the solution will be:

$$i = A_1e^{(-\alpha+jg)t} + A_2e^{(-\alpha-jg)t}. \qquad (129)$$

We could, of course, proceed to evaluate A_1 and A_2 from initial conditions, similarly to the way in which this was done for case I above. But if we compare (129) with (118), we see that the two are the same except that we now have jg in place of β. Hence we can obtain the solution for this case by replacing β by jg in eq. (122) which was the final result for case I. Thus we get:

$$i = \frac{Ee^{-\alpha t}}{jLg}\sinh jgt.$$

But:

$$\sinh jgt = j\sin gt$$

so that:

$$i = \frac{E}{Lg} e^{-\alpha t} \sin gt. \tag{130}$$

It is interesting to compare this result with that for the L, C case given by eq. (95). We can easily see that (130) will go over into (95) for $R = 0$, because for the limit:

$$\left.\begin{aligned} R &\to 0 \\ \alpha &\to 0 \\ g &\to g_0. \end{aligned}\right\} \tag{131}$$

A plot of eq. (130) is shown in Fig. 23. Comparing this with Fig. 14 for the case without resistance, we see that the outstanding difference is that now the amplitude decreases as time increases instead of remaining constant. This decay is due to the factor $e^{-\alpha t}$ in the expression (130). The function:

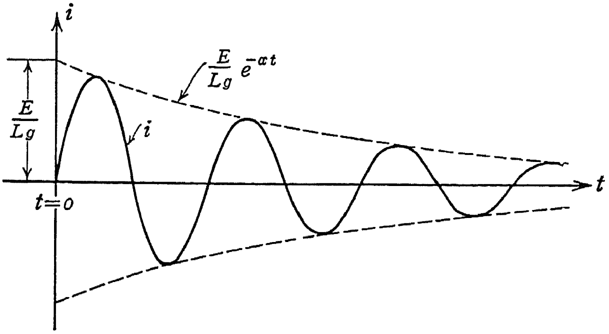


FIG. 23

$$\frac{E}{Lg} e^{-\alpha t}$$

acts as an envelope for the sine function. We say that the oscillation now decays exponentially. The rate of decay is regulated by the factor α. This factor is therefore called the **damping constant** or the **decrement** of the circuit. It is the reciprocal of the time constant T. The latter we remember is that time which it takes for the amplitude to diminish to $1/e$ of its initial value. Thus:

$$T = \frac{1}{\alpha} = \frac{2L}{R}. \tag{132}$$

Note that this time constant is just twice that for the simple R, L circuit given by eq. (69a).

We see therefore that if we suddenly impress a constant e.m.f. to an R, L, C circuit, we obtain an oscillatory surge which continues for an increasing length of time the smaller the resistance in the circuit becomes, and finally continues forever in the limit $R \rightarrow 0$. For very large values of R, no oscillation takes place at all, as shown by case I above. This case is called the **aperiodic** case, and consists merely of a surge. The surge lasts longer the larger R becomes. Case II forms the boundary between the aperiodic and the oscillatory cases. It is called the **critically damped** case. It still consists merely of a surge of current, but the average time the surge lasts is less than that of any aperiodic case.

A fairly good physical picture of this whole situation may be had if we consider the ordinary pendulum as a mechanical analogy of the electrical oscillatory system. Suppose we suspend the pendulum in a tank filled with heavy molasses. If we pull the bob out and let it go it will slowly ooze back to its rest position and stay there. The friction in the molasses is too high to allow oscillation. This is the aperiodic case. As we thin out the molasses the friction decreases. Hence the bob will come back to the vertical position faster. As we continue the thinning-out process, we will come to a point where the bob just fails to overshoot the vertical position on its return. This will be the case of swiftest descent, and represents the critically damped case. If we thin the molasses still more the pendulum becomes oscillatory. Finally if we could remove all friction, the pendulum would swing forever.

Getting back to our last case again, let us calculate and plot the expressions for the inductance, the resistance and the capacitance voltages. We have:

$$e_i = L\frac{di}{dt}.$$

Substituting (130) into this, we get:

$$e_i = \frac{E}{g}e^{-\alpha t}(g\cos gt - \alpha \sin gt).$$

Noting the first eq. (128), this may be converted into:

$$\left.\begin{aligned} e_i &= E\cdot\frac{g_0}{g}\cdot e^{-\alpha t}\cdot\cos(gt+\gamma) \\ \text{where:}\qquad \gamma &= \tan^{-1}\left(\frac{\alpha}{g}\right). \end{aligned}\right\} \tag{133}$$

The resistance voltage or Ri drop is simply R times eq. (130). In terms of α this may be written:

$$e_r = 2E\cdot\frac{\alpha}{g}\cdot e^{-\alpha t}\cdot\sin gt. \tag{134}$$

The capacitance voltage is given by:

$$e_c = \frac{1}{C}\int i\,dt. \tag{135}$$

Substituting (130) into this expression and performing the integration, we have:

$$e_c = -E\cdot\frac{g_0}{g}\cdot e^{-\alpha t}\cdot\cos(gt-\gamma) + \text{const.} \tag{136}$$

The constant in (136) is evaluated from the condition that: $e_c = 0$ for $t = 0$, so that:

$$\text{const.} = E\cdot\frac{g_0}{g}\cos\gamma = E \tag{137}$$

and the final expression becomes:

$$e_c = E\left[1 - \frac{g_0}{g}\cdot e^{-\alpha t}\cdot\cos(gt-\gamma)\right]. \tag{138}$$

The student should evaluate the integration indicated by (135) himself in order to appreciate the amount of manipulation involved. The integrand is composed of the product of the two functions $e^{-\alpha t}$ and $\sin gt$. A repeated application of the formula for integration by parts is necessary to carry this out. Later we shall give a fuller demonstration of the use of complex notation for the representation of harmonic functions, by means of which such operations as the one above become very simple. We prefer to have the reader follow this cumbersome method for the time being so that he will appreciate the complex method all the more by comparison.

Incidentally, it is well to note a very simple relationship between the damped and undamped angular frequencies g and g_0, the damping constant α, and the phase angle γ. This relation is very useful generally, and particularly in the operations involved above for finding e_i and e_c. This relation is expressed by the first equation (128) which we repeat here for convenience:

$$g = \sqrt{g_0^2 - \alpha^2}.$$

This represents a right triangle as shown in Fig. 24. α and g are the legs of this triangle, and g_0 is the hypotenuse. Usually α is small compared to either g or g_0, which means a highly oscillatory circuit. Then g and g_0 are very nearly alike, and the angle γ is very small. For vanishing resistance, $\alpha = 0$, and the damped and undamped frequencies coincide. This figure makes the relationships between these quantities stand out more clearly than the analytical expressions do.

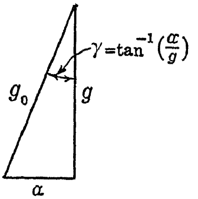


FIG. 24

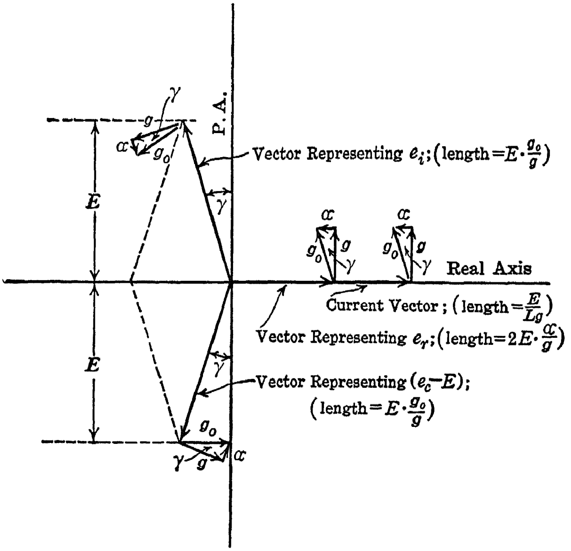

γ
g
α
g_0
P. A.
Vector Representing e_i; (length $= E \cdot \frac{g_0}{g}$)
γ
E
α
α
g_0
g
γ
g_0
g
γ
Real Axis
Current Vector; (length $= \frac{E}{Lg}$)
Vector Representing e_r; (length $= 2E \cdot \frac{\alpha}{g}$)
E
γ
Vector Representing $(e_c - E)$; (length $= E \cdot \frac{g_0}{g}$)
g_0
γ
g
α

FIG. 25

Another aid in correlating the relationships between the current and the inductance, resistance, and capacitance voltages is a vector diagram. This is a representation of these quantities as vectors in the complex plane. The instantaneous values are thought of as projections of these vectors upon some projection axis. We shall use the vertical or imaginary axis for this purpose. The vectors represent the respective quantities at the instant $t = 0$. Such a diagram is shown in Fig. 25. Instead of representing the condenser voltage directly, we have drawn in the vector $e_c - E$, which is the variable portion of that voltage. The actual condenser voltage is this variable part plus the constant value E. The arrows g_0 at the tips of all the vectors indicate that they are about to move in that direction. This direction is not normal to the vectors, but inward. In other words the tips of the vectors follow a logarithmic spiral instead of a circle, and the directions g_0 shown in Fig. 25 are tangent to the respective spirals at their origin. These directions are off the normal by the angle γ. It will be noticed that the three voltage vectors are in equilibrium, i.e., their vector sum is zero. Thus:

$$e_i + (e_c - E) + e_r = 0$$

or:

$$e_i + e_c + e_r = E \tag{139}$$

as it should be.

This vector representation of exponentially decaying harmonic functions is extremely useful since it gives us a complete picture of the entire situation regarding relative magnitudes, phase positions, and voltage balance, as well as a good idea of the rate of decay of the vectors. It carries still farther the relation between α, g, and g_0, and the effect of damping upon the natural angular frequency. Without the presence of resistance, the linear velocity at which the tips of the vectors move is $g_0 = \frac{1}{\sqrt{LC}}$ times the vector-length, and the direction in which they move is at right angles to the vector. For a given amount of damping, characterized by α, the velocity at which the tip of any vector moves is still the same g_0 times the vector-length, but the direction is now off the normal by the angle γ. This means that the linear velocity of a vector-tip must now be split into two components: One normal to the direction of the vector, and one opposite to the vector direction. The normal component is g, or the damped angular velocity, times the vector-length, whereas the opposing or inward component is α, or the exponential rate of decay or shrinkage, times the vector-length. Since the net velocity of the vector tip (in cms. per sec.) has not changed due to the introduction of resistance, but still is g_0 times the vector-length, the component proportional to g must be less than that proportional to g_0. The vector actually rotates more slowly due to the introduction of resistance, but in addition it now has a rate of shrinkage proportional to α such that the components α plus g vectorially give the same g_0, as indicated in the figure.

A vector diagram is nothing more than a polar plot in the complex plane. The vectors themselves at any instant show the magnitudes as well as the relative phases of the respective quantities which they represent. The instantaneous values are given by the projections of these vectors upon an arbitrarily chosen reference line. In the above, the vertical axis serves this purpose. It is interesting to note that in this diagram of Fig. 25 the inductance-voltage leads the current by more than 90°, while the capacitance-voltage lags by more than 90°. This condition is quite different from that encountered in steady state alternating current performance. There these angles can never exceed 90°. In the case of natural oscillations such as these however, it is quite possible for a current to lag or lead an induced voltage by more than 90°.

Since this vector representation facilitates the graphical understanding of this problem, it ought to do the same for the analytical work. This is a point which we wish to emphasize now. From Fig. 25 we could write for the current in our R, L, C circuit under the influence of a constant driving force:

$$i = \mathcal{I}_m[Ie^{mt}] \tag{140}$$

where:

$$I = \frac{E}{Lg}\underline{|0^\circ} \tag{141}$$

and:

$$m = -\alpha + jg = g_0\underline{\left|\gamma + \frac{\pi}{2}\right.}\cdot \tag{142}$$

Then:

$$\begin{aligned} e_i &= L\frac{di}{dt} \\ &= L\frac{d}{dt}\cdot \mathcal{I}_m[Ie^{mt}] \\ &= \mathcal{I}_m[LIme^{mt}] \\ &= \mathcal{I}_m[E_ie^{mt}] \end{aligned} \tag{143}$$

where:

$$E_i = Lg_0 I \left|\underline{\gamma + \frac{\pi}{2}}\right. = E \cdot \frac{g_0}{g} \left|\underline{\gamma + \frac{\pi}{2}}\right. \cdot \tag{144}$$

This result checks the Fig. 25 exactly and hence also the expression (133) as the student may demonstrate for himself. For the condenser-voltage we now have:

$$\begin{aligned} e_c &= \frac{1}{C}\int \mathcal{I}_m[Ie^{mt}] \\ &= \mathcal{I}_m\left[\frac{I}{Cm}e^{mt}\right] + \text{const.} \\ &= \mathcal{I}_m\,[E_c e^{mt}] + \text{const.} \end{aligned} \tag{145}$$

where:

$$\begin{aligned} E_c &= \frac{I}{Cg_0}\left|\overline{\gamma + \frac{\pi}{2}}\right. = \frac{E}{LgCg_0}\left|\overline{\gamma + \frac{\pi}{2}}\right. \\ &= E \cdot \frac{g_0}{g}\left|\overline{\gamma + \frac{\pi}{2}}\right. \cdot \end{aligned} \tag{146}$$

On account of $e_c = 0$, for $t = 0$, the constant evaluates to:

$$-\mathcal{I}_m[E_c] = E$$

so that:

$$(e_c - E) = \mathcal{I}_m[E_c e^{mt}]. \tag{147}$$

This result also checks the Fig. 25. The reader should note the ease with which this last integration was carried out as compared to the evaluation of (135) in obtaining (136).

The complex representation of a harmonic function is a little difficult to grasp, but it has so many advantages over the representation by means of sines and cosines with phase angles that it is certainly worth spending some time on. This will be appreciated still more when we take up alternating impressed forces. There the complex notation will be adopted altogether. Sines and cosines will be used as sparingly as possible throughout this text. The student should practice the visualization of harmonic functions in terms of rotating vectors in the complex plane. In order to show how these are linked graphically, we give the Fig. 26 in which the complete polar time plot for Fig. 25 is given together with the corresponding rectangular representation. The student should note in the rectangular plot that the instantaneous sum of e_i, e_r, and e_c is a constant and equal to E for all values of time. This is not so easy to see in the rectangular plot, while it is evident at a glance from the polar plot because there we see that the vectors E_i, E_r, and E_c add to zero. The rectangular plot of e_c is interesting and should be compared with the corresponding plot for the case without damping shown in Fig. 15. Due to the damping action we now see that the condenser voltage settles down to the constant value E as it eventually should.

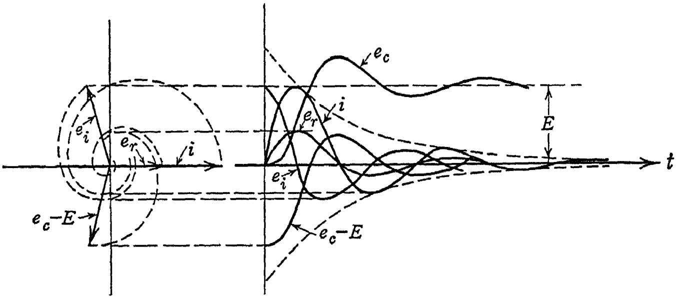


FIG. 26

6. The Vector Treatment for the R, L, C Circuit. It is quite simple to apply the vector method of representation to the R, L, C circuit, just as this was done for the L, C circuit, the result in that case being given by eq. (110) and illustrated by Fig. 19. This vector method is useful for determining the condition of oscillation in the circuit after the applied force has suddenly changed, when the circuit is in the process of oscillation at the time the change occurs. We refer again to Fig. 20 for the switching arrangement.

Let the switch be in position 2 with the circuit in oscillation. The current may then be represented by:

$$i_b = \mathcal{I}_m[Be^{mt}] \tag{148}$$

where B is a complex amplitude and m is the complex angular velocity given by eq. (142). The only difference between this and the form for the current in the L, C case (eq. 103a) is that jg_0 is replaced by m.

At a time which we designate by $t = 0$ for convenience, the switch is suddenly thrown to the position 1. Suppose this produces a change in the impressed force which may be represented by ΔE, thus leaving the magnitude and direction of this sudden change arbitrary. After $t = 0$ the oscillation has the same form and can be represented by:

$$i_a = \mathcal{I}_m[Ae^{mt}]. \tag{149}$$

At the switching instant we have to satisfy two conditions. The continuity of current is expressed by:

$$(i_a)_{t=0} = (i_b)_{t=0} \tag{150}$$

and the voltage balance requires that:

$$\left(\frac{di_a}{dt}\right)_{t=0} - \left(\frac{di_b}{dt}\right)_{t=0} = \frac{\Delta E}{L}. \tag{151}$$

Applying (150) to (148) and (149), gives:

$$\mathcal{I}_m[A] = \mathcal{I}_m[B]$$

or:

$$A_2 = B_2 \tag{152}$$

just as in the case of the L, C circuit. Applying (151), we get:

$$\mathcal{I}_m[Am - Bm] = \frac{\Delta E}{L}$$

so that:

$$\mathcal{I}_m[(A_1 + jA_2 - B_1 - jB_2)\,(-\alpha + jg)] = \frac{\Delta E}{L}$$

or:

$$(A_1 - B_1)g + (A_2 - B_2)\alpha = \frac{\Delta E}{L}.$$

Noting (152), we have:

$$A_1 = B_1 + \frac{\Delta E}{Lg}. \tag{153}$$

Finally, multiplying (152) by j and adding it to (153), we get:

$$A = B + \frac{\Delta E}{Lg} \tag{154}$$

which is exactly the result (110) except that g_0 is replaced by g.

In order to illustrate this result, let us suppose that the R, L, C circuit of Fig. 20 is at rest and the switch is in position 2. The following switching operations are now carried out:

Time	*Switch-Position*
$t < 0$	2
$t = 0$	1
$t = \frac{3\pi}{g}$	2
$t > \frac{3\pi}{g}$	2

It is desired to know the complete solution after $t = 0$. We have to consider two switching operations, to each of which the relation (154) is applied with the proper significance of A, B, and ΔE. For the operation at $t = 0$, $B = 0$ because the circuit condition before $t = 0$ was a rest condition. ΔE is $+E$, i.e., the battery voltage is applied in the positive assumed direction. Hence:

$$A = \frac{E}{Lg}.$$

For the interval $0 < t < \frac{3\pi}{g}$ therefore, the solution is:

$$i = \mathcal{I}_m\left[\frac{E}{Lg}e^{mt}\right]. \tag{155}$$

For the switching operation at $t = \frac{3\pi}{g}$, B is not zero, but is equal to the complex expression within the brackets of eq. (155) for that value of t. Thus:

$$B = \frac{E}{Lg}e^{\frac{3m\pi}{g}}.$$

But:

$$e^{\frac{3m\pi}{g}} = e^{-\frac{3\alpha\pi}{g}} \cdot e^{j3\pi} = -e^{-\frac{3\alpha\pi}{g}}$$

so that:

$$B = -\frac{Ee^{-\frac{3\alpha\pi}{g}}}{Lg}. \tag{156}$$

At this instant the battery is suddenly removed from the circuit. This means that $\Delta E = -E$. The application of (154) for this switching operation therefore gives:

$$\begin{aligned} A &= -\frac{Ee^{-\frac{3\alpha\pi}{g}}}{Lg} - \frac{E}{Lg} \\ &= -\frac{E}{Lg}\left(1 + e^{-\frac{3\alpha\pi}{g}}\right). \end{aligned} \tag{157}$$

Hence the solution for the interval $t > \frac{3\pi}{g}$ is given by:

$$i = \mathcal{I}_m\left[-\frac{E}{Lg}\left(1 + e^{-\frac{3\alpha\pi}{g}}\right)e^{m\left(t-\frac{3\pi}{g}\right)}\right]. \tag{158}$$

The results (155) and (158) are illustrated in Fig. 27. On the left of this figure is the vector representation or polar plot of the rotating and decaying vectors. On the right is the corresponding rectangular plot showing the damped harmonic functions in their usual form. The interval from $t = 0$ to $t = \frac{3\pi}{g}$ corresponds to one and one-half cycles. At $t = 0$ the vector $A = \frac{E}{Lg}$ is created, and immediately starts to rotate and decay according to the natural mode of the circuit. At the instant $t = \frac{3\pi}{g}$, this vector has shrunk considerably and is in the negative horizontal position. The sudden removal of the battery introduces the negative real vector of magnitude $\frac{E}{Lg}$, the same size as that originally created, so that the resultant vector just after $t = \frac{3\pi}{g}$ is larger than the

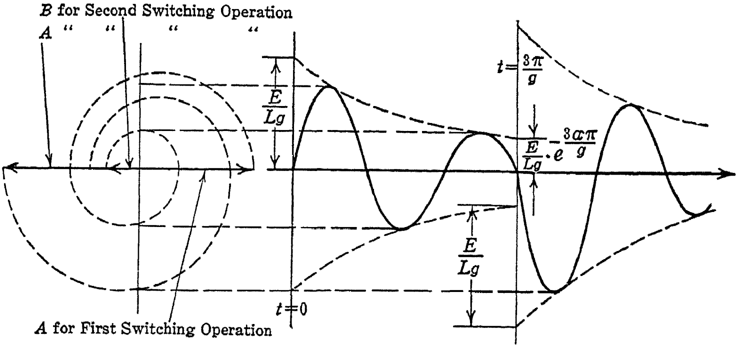

B for Second Switching Operation
A " " " "
$\frac{E}{Lg}$
$t=\frac{3\pi}{g}$
$\frac{E}{Lg}\cdot e^{-\frac{3\alpha\pi}{g}}$
t
$\frac{E}{Lg}$
t=0
A for First Switching Operation

Fig. 27

original one. This is the A of the second switching operation. It immediately rotates and decays at its normal rate. In the rectangular plot these occurrences are illustrated by means of harmonic functions and exponential envelopes.

This method of attack on the R, L, C circuit may be used in various problems. If an applied voltage varies in an erratic manner, it may be broken up into steps sufficiently small to give the required accuracy. Each step may be treated in the above manner. Another application is the oscillatory circuit which is excited by means of repeated applications and removals of a substantially constant e.m.f. It can be seen from the result shown in Fig. 27 that it is possible to produce sustained oscillations by repeatedly applying and removing a constant voltage. This fundamental principle takes many interesting forms in various schemes that have been proposed for the production of continuous oscillations. The latter term is, of course, not rigorously applied in this case because the oscillations so produced are not continuous, but consist of a succession of damped sinusoids. Although this kind of a wave form may be used for radio telegraphy, it is of no use in radio telephony. On the other hand, for the production of high frequencies to be used in induction furnaces, such methods, as the above are used, since wave form is not an essential requirement.

PROBLEMS TO CHAPTER II

2–1. A force-function defined by:

$$\begin{aligned} f(t) &= 0 \quad \text{for } t < 0 \\ f(t) &= 10 \text{ for } 0 < t < 2\cdot 10^{-4} \\ f(t) &= 5 \quad \text{for } 2\cdot 10^{-4} < t < 3\cdot 10^{-4} \\ f(t) &= 10 \text{ for } 3\cdot 10^{-4} < t < 4\cdot 10^{-4} \\ f(t) &= 0 \quad \text{for } t > 4\cdot 10^{-4} \end{aligned}$$

is impressed upon an R, L circuit with:

$$L = 10^{-3} \qquad R = 5.$$

The initial conditions are given by:

$$i = 0 \quad \text{for} \quad t = 0.$$

Determine and plot the complete current solution. Also plot the kinetic and loss energies as functions of time.

2–2. The same force-function as in Prob. **2–1** is impressed upon an R, C circuit with:

$$R = 5 \qquad C = 4\cdot 10^{-5}.$$

Determine and plot the complete current solution assuming initial rest conditions. Plot also the condenser charge and the condenser voltage as well as the potential and loss functions.

2–3. The same force-function as in Prob. **2–1** is impressed upon an L, C circuit with:

$$L = \frac{10^{-3}}{\pi} \qquad C = \frac{10^{-5}}{\pi}.$$

Assuming initial rest conditions, determine the current, charge, inductance-voltage, capacitance-voltage, kinetic energy, potential energy, and total energy, and plot these as functions of time.

2–4. Repeat Prob. **2–3** including a resistance:

$$R = \frac{20}{\pi}.$$

In this case also determine the loss energy as a function of time.

2–5. An L, C circuit with the same data as in Prob. **2–3** is initially at rest. At $t = 0$ an e.m.f. of 10 volts is suddenly impressed. At $t = 2 \cdot 10^{-4}$, half of the inductance is suddenly shorted out. Determine and plot the complete current solution. Repeat for the condition that half the inductance is suddenly shorted out at $t = 2.5 \cdot 10^{-4}$ seconds.

CHAPTER III

THE SINGLE MESH CIRCUIT WITH HARMONIC EXCITATION

1. Preamble Concerning Complex Notation. The student is no doubt somewhat familiar with the use of complex expressions for the representation of harmonic functions. We have made use of this device to some extent in the vector methods given in the last chapter. In alternating current work the use of complex quantities is more fully appreciated than in any other field. In fact the complex notation is almost indispensable in the treatment of networks with alternating e.m.f's. It is therefore quite proper that we should spend a few pages here in order to fix ideas more thoroughly.

Suppose we consider a harmonic generator having a frequency of f cycles per second and hence an angular frequency of $\omega = 2\pi f$ radians per second. It is possible to represent the generated voltage of this machine by:

$$e(t) = V \sin(\omega t + \chi) \tag{159}$$

where V is the maximum value and χ is an arbitrary phase angle which fixes the value of e at the start $t = 0$. Both sides of (159) are real functions of t. We may also represent the generated voltage of this machine by the analytic expression:

$$e(t) = Ee^{j\omega t} \tag{160}$$

where E is complex and e is the Napierian base. Here we should note something peculiar right away. The left-hand side of (160) is real whereas the right-hand side is complex. Evidently something is omitted. This is the case. The right-hand side represents a vector E which starts from some initial position for $t = 0$ and rotates with a constant angular velocity of ω radians per second in the counter-clockwise direction. This is indicated in Fig. 28. If we observe the projection of this vector upon any stationary axis we note that this projection performs harmonic motion and therefore represents a harmonic function. It is the real projection of the right-hand side of (160) upon some reference axis that is meant to be understood when this form is written in place of the eq. (159). It remains to show why it is justifiable to omit this operation of projection.

Any axis may be used for purposes of projection. Let us choose the real or horizontal axis. Then (160) should be written:

$$e(t) = \mathcal{R}_e\,[Ee^{j\omega t}] \tag{160a}$$

where $\mathcal{R}_e$ is an operator which takes the projection of the function following it, upon the real axis. Another way of expressing this

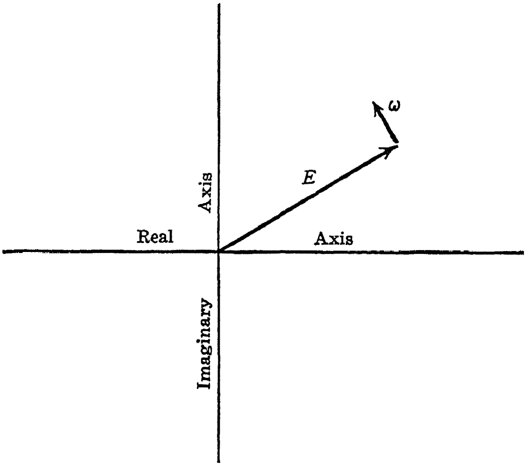


FIG. 28

is to say that $\mathcal{R}_e$ is an operator which takes only the real portion of the function it operates upon. With this operation fixed let us determine E so that (160a) will become identical with (159). For this purpose suppose we let:

$$E = |E|\,\underline{|\psi}. \tag{161}$$

Then (160a) becomes:

$$\begin{aligned} e(t) &= \mathcal{R}_e\,[\,|E|\,e^{j(\omega t+\psi)}] \\ &= |E| \cos(\omega t + \psi) \\ &= |E| \sin\left(\omega t + \psi + \frac{\pi}{2}\right). \end{aligned} \tag{162}$$

Comparing (159) and (162), we see that:

$$\left.\begin{aligned}|E| &= V\\ \psi &= \chi - \frac{\pi}{2}.\end{aligned}\right\}\qquad(163)$$

Hence:

$$E = V\,\underline{\big|\chi - \frac{\pi}{2}}.\qquad(164)$$

This is illustrated by Fig. 29. The value of the voltage for $t = 0$ is $e(0)$, the projection of E upon the real axis. This figure represents a sine function advanced in phase by χ radians or a cosine function retarded in phase by ψ radians. Note also that this function (159) or its equivalent (160a) might be written:

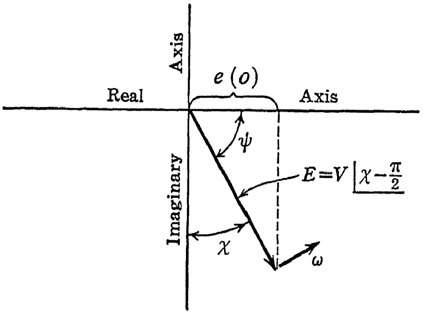


FIG. 29

$$e(t) = V\cos\chi\sin\omega t + V\sin\chi\cos\omega t\qquad(165)$$

or:

$$e(t) = |E|\sin\psi\sin\omega t + |E|\cos\psi\cos\omega t.\qquad(165a)$$

Hence if the voltage were given by:

$$e(t) = a\sin\omega t + b\cos\omega t\qquad(166)$$

it would follow that:

$$\left.\begin{aligned}a &= V\cos\chi = |E|\sin\psi\\ b &= V\sin\chi = |E|\cos\psi\\ \chi &= \tan^{-1}\left(\frac{b}{a}\right)\\ \psi &= \tan^{-1}\left(\frac{a}{b}\right).\end{aligned}\right\}\qquad(167)$$

A comparison of the above relations shows that a harmonic function, having an arbitrary amplitude and phase, may be represented analytically in any of the following ways:

By means of a sine with a phase angle,

by means of a cosine with a phase angle,

by means of a sine plus a cosine term,
by means of the real portion of the product of a complex amplitude and an exponential function.

Any one of these forms can be converted into any other. Also note that the last named method may be varied in that the rotating vector need not necessarily be projected upon the real axis, although we shall adopt this convention in all the following work.

It may seem that the complex way of writing the harmonic function is more cumbersome than any other. This is not the case for the following reason. The usual operations which we have to perform in network analysis are:

$$\left.\begin{array}{l}\text{differentiation}\\ \text{integration}\\ \text{addition and subtraction.}\end{array}\right\} \qquad (168)$$

Now we can easily see that any of these operations are commutative with respect to the operation of taking the real portion, when operating upon a complex function. By this we mean that if $f(t)$ represents any complex function, and the operator O stands for any one of the above-named operations, then:

$$O \cdot \mathcal{R}_e \cdot [f(t)] = \mathcal{R}_e \cdot O \cdot [f(t)]. \qquad (169)$$

For example: The real part of the derivative of $f(t)$ is the derivative of the real part, etc.

On the basis of this reasoning we can draw the following conclusions. The functions with which we deal in network theory have the form (160a). (Current would be written in the same complex form as voltage.) The operations that we are called upon to perform are those given in (168). Since these operations are commutative with respect to the $\mathcal{R}_e$-operator, we may drop the $\mathcal{R}_e$-sign in (160a) and use the simpler form (160) while manipulating the function according to the requirements of the differential equations involved. After the process of integration is over, we again introduce the $\mathcal{R}_e$-operator before interpreting the result. This omission of the $\mathcal{R}_e$-operator during the process of integration saves much time and space. It must not be forgotten, however, that the $\mathcal{R}_e$-sign must be inserted before the final result can be interpreted in terms of instantaneous values.

By this method of procedure we have accomplished a remarkable saving in time and effort. Any harmonic function may be represented by the single expression (160). The question of phase angles — which is an eternal nuisance when the form (159) is used — takes care of itself automatically. The phase angle of the function does not even appear in the form (160). It is hidden in the complex character of E, and therefore is out of the way during the process of integration when it is not needed anyway. Furthermore, the only function we have left is the exponential, which is so easy to differentiate or integrate. In the following we shall endeavor to illustrate these points.

Before proceeding we wish to call attention to the following precaution. It should always be remembered in the use of exponential forms that the legitimate operations do not include multiplication and division. These are not commutative with respect to the $\mathcal{R}_e$-operator. For example, the real portion of a product of two complex functions is not equal to the product of their real portions, i.e., if:

$$a(t) + j\,b(t) = f_1(t)$$

and:

$$c(t) + j\,d(t) = f_2(t)$$

are the two functions, then:

$$\mathcal{R}_e[f_1(t)] \cdot \mathcal{R}_e[f_2(t)] = a(t) \cdot c(t)$$

whereas:

$$\mathcal{R}_e[f_1(t) \cdot f_2(t)] = a(t) \cdot c(t) - b(t) \cdot d(t).$$

Such instances do occur in network problems. For example, the power is expressed by the product of voltage and current. If these are given by:

$$e(t) = \mathcal{R}_e[Ee^{j\omega t}]$$

and:

$$i(t) = \mathcal{R}_e[Ie^{j\omega t}]$$

then it is *not* true that:

$$\text{Instantaneous power} = \mathcal{R}_e[EIe^{j2\omega t}].$$

The student should guard himself against making such mistakes as these.

The correct procedure to find the power is as follows. We can omit the $\Re_e$ signs and write:

$$e(t) = \tfrac{1}{2}\,(Ee^{j\omega t} + \bar{E}e^{-j\omega t})$$

and:

$$i(t) = \tfrac{1}{2}\,(Ie^{j\omega t} + \bar{I}e^{-j\omega t})$$

where the bars indicate the conjugate value. Then the instantaneous power is given by:

$$e \cdot i = \tfrac{1}{4}\,(EIe^{j2\omega t} + \bar{E}\bar{I}e^{-j2\omega t} + \bar{E}I + E\bar{I}).$$

The average power per cycle is given by the integral:

$$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} e \cdot i\,dt = \tfrac{1}{4}\,(\bar{E}I + E\bar{I})$$

as the reader may verify by carrying out the integration.

2. The R, L Circuit. We shall now consider again the simple circuit treated in Chapter II, § 1, but this time with an impressed voltage which is alternating instead of direct. This is illustrated in Fig. 30. Here $e(t)$ represents a harmonic voltage of constant frequency. The switching arrangement is the same as that used previously. In accordance with our above discussion, we may write:

$$e(t) = Ee^{j\omega t} \tag{170}$$

where ω is the angular frequency, and the $\Re_e$-operation is tacitly understood. At a time $t = 0$, the switch S is thrown from position 2 to position 1. Then the differential equation expressing dynamic equilibrium between impressed and counter-forces for $t \geqq 0$, becomes:

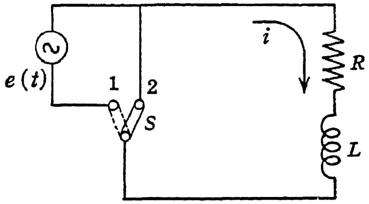


FIG. 30

$$L\frac{di}{dt} + Ri = Ee^{j\omega t}. \tag{171}$$

Let us find the particular integral or steady state first. Here it is logical to assume that:

$$i_s = Ie^{j\omega t} \tag{172}$$

where again the $\mathcal{R}_e$-operation is tacitly assumed, and I is a constant complex amplitude which will automatically take care of the relative phase between the voltage and the current in the steady state. Substituting the assumption (172) into the circuit equation (171), we obtain:

$$(Lj\omega + R)Ie^{j\omega t} = Ee^{j\omega t} \tag{173}$$

which is the condition under which (172) will be an integral of (171). This condition is easily met by the relation:

$$I = \frac{E}{R + jL\omega} \tag{174}$$

which determines the unknown complex amplitude in (172).

Thus our steady-state current becomes:

$$i_s = \mathcal{R}_e\left[\frac{Ee^{j\omega t}}{R + jL\omega}\right]. \tag{175}$$

The denominator of this expression is the complex impedance of the circuit to the frequency ω, thus:

$$\left.\begin{aligned} Z &= R + jL\omega \\ &= |Z|\,\underline{|\theta} \end{aligned}\right\} \tag{176}$$

so that:

$$\left.\begin{aligned} |Z| &= \sqrt{R^2 + L^2\omega^2} \\ \text{and:}\qquad \theta &= \tan^{-1}\left(\frac{L\omega}{R}\right). \end{aligned}\right\} \tag{177}$$

At the instant $t = 0$, $e(t)$ may have any arbitrary phase, so that:

$$E = |E|\,\underline{|\psi}. \tag{178}$$

If ψ is zero, then $e(t)$ is a cosine function. If ψ is $-\frac{\pi}{2}$, then $e(t)$ is a sine function. For any intermediate value of ψ, $e(t)$ represents either a sine or a cosine with a proper phase angle. It is not necessary for the present discussion to limit ourselves to any particular value of ψ. This may be done later to conform with some particular case. By means of (176), (177), and (178), we may write the result (175) in the form:

$$i_s = \mathcal{R}_e\left[\left|\frac{E}{Z}\right|e^{j(\omega t + \psi - \theta)}\right] \tag{179}$$

or:

$$i_s = \left|\frac{E}{Z}\right| \mathcal{R}_e\left[e^{j(\omega t+\psi-\theta)}\right] \tag{179a}$$

or:

$$i_s = \left|\frac{E}{Z}\right| \cos(\omega t + \psi - \theta). \tag{179b}$$

Now since the voltage may be written in the form:

$$e(t) = |E| \cos(\omega t + \psi) \tag{170a}$$

it follows that the steady-state current is a harmonic function, whose amplitude is the magnitude of the quotient of voltage and impedance, lagging behind the voltage function by the angle of the impedance θ. The vector representation of this is given in Fig. 31, while the rectangular plot is shown in Fig. 32. The scales for voltage and current are arbitrary. The resultant phase angle for the current is:

$$\phi = \psi - \theta \tag{180}$$

as may be seen from eq. (179b) or from the vector diagram of Fig. 31. Since θ is here taken greater than ψ, ϕ comes out negative, which puts I below the reference axis in Fig. 31. In Fig. 32 we see that i_s is a cosine function retarded by the time $\frac{\phi}{\omega}$ which checks the negativeness of ϕ.

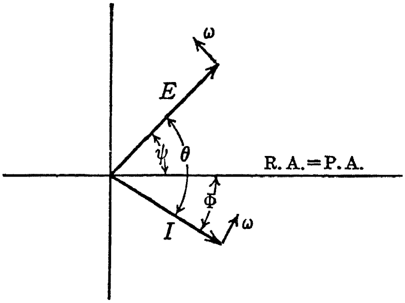


FIG. 31

Incidentally we see that the magnitude of the steady-state current from (179b) is simply:

$$|I| = \left|\frac{E}{Z}\right|. \tag{181}$$

If we imagine this equation divided through by $\sqrt{2}$, then the voltage and current amplitudes will be root-mean-square instead of maximum values. Hence if we are not interested in relative phase in the steady state, we need only take the magnitude of the quotient of voltage and impedance. The current will be r.m.s., if the voltage is r.m.s. In general the current lags behind the voltage by the impedance angle. In the present case θ is a positive angle as given by (177). Hence the current actually lags. In general the impedance angle may also be negative. A current which lags by a negative angle actually leads by a positive angle. We shall come to this case in the next section.

Having found the steady-state solution to our problem, we continue with the determination of the transient portion of the complete solution. This is the solution of the force-free equilibrium condition corresponding to eq. (171). This is obviously:

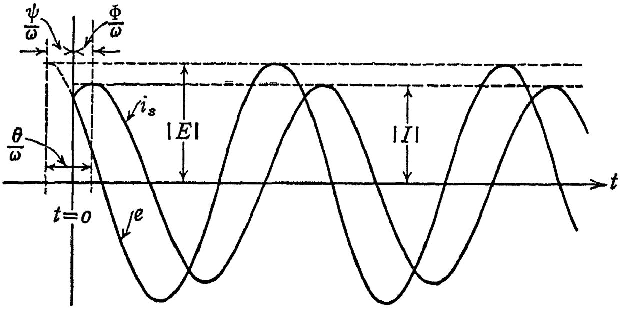


FIG. 32

$$L\frac{di}{dt} + Ri = 0. \tag{182}$$

But this is, of course, the same as the force-free equation which we considered while obtaining the solution for the R, L circuit with constant excitation in Chapter II, § 1. Hence the transient solution does not differ in form with various kinds of impressed forces. The transient solution for this case is given by eq. (65) of the last chapter. We repeat it here for convenience:

$$i_t = Ae^{-\frac{Rt}{L}} \tag{183}$$

where:

$$-\frac{R}{L}$$

is the natural mode of the circuit, and A is the integration constant. Notice that although (183) is the transient solution for the R, L circuit for any kind of an impressed force, the value of the integration constant A is different for different kinds of e.m.f's. We proceed to evaluate A for this case.

The complete solution is now:

$$i = \mathcal{R}_e[Ie^{j\omega t}] + Ae^{-\frac{Rt}{L}} \tag{184}$$

where I is given by (174). Suppose the one initial condition is given in the form:

$$i = i_0 \quad \text{for} \quad t = 0. \tag{185}$$

Substituting this into (184), gives:

$$i_0 = \mathcal{R}_e[I] + A$$

or:

$$A = i_0 - \mathcal{R}_e[I]. \tag{186}$$

Hence the general solution for this case becomes:

$$i = \mathcal{R}_e[Ie^{j\omega t}] + \{i_0 - \mathcal{R}_e[I]\}e^{-\frac{Rt}{L}}. \tag{187}$$

If we write:

$$I = I_1 + jI_2 \tag{188}$$

then:

$$i = \mathcal{R}_e[Ie^{j\omega t}] + (i_0 - I_1)e^{-\frac{Rt}{L}} \tag{189}$$

or:

$$i = |I| \cos(\omega t + \phi) + (i_0 - I_1)e^{-\frac{Rt}{L}}. \tag{189a}$$

Particularly for $i_0 = 0$ we get:

$$i = \mathcal{R}_e[Ie^{j\omega t}] - I_1e^{-\frac{Rt}{L}}. \tag{190}$$

Note that I_1 is the instantaneous value of the steady state at $t = 0$. Hence we see from eq. (189) that the transient term will be absent if $i_0 = I_1$, i.e., if the current flowing in the circuit at the switching instant equals that demanded by the steady state at that instant. The transient appears whenever there is an initial discrepancy, and the magnitude of the transient equals this discrepancy. In eq. (190) the available current in the circuit at $t = 0$ is assumed zero. Even then it is possible for the transient to be absent, namely then when $I_1 = 0$. Whether or not this is so depends upon the phase of the voltage at $t = 0$. The situation shown in Fig. 31 is far from this condition. But if the voltage phase ψ had been less by $\psi + \frac{\pi}{2} - \theta$ radians, i.e., if it had been $\theta - \frac{\pi}{2}$, then I would have been a vector straight down, and its real portion would have been zero. Consequently (190) would contain no transient.

We can determine that voltage phase which will make the transient vanish in the general case. The condition is that:

$$i_0 = I_1.$$

Using eq. (174), this is:

$$i_0 = \mathcal{R}e\left[\frac{E}{R + jL\omega}\right]$$

or by (177) and (178):

$$i_0 = \mathcal{R}e\left[\left|\frac{E}{Z}\right| \underline{|\psi - \theta}\right]$$

or:

$$i_0 = \left|\frac{E}{Z}\right| \cos(\psi - \theta). \tag{191}$$

Solving for ψ, we have:

$$\psi = \theta + \cos^{-1}\left(i_0\left|\frac{Z}{E}\right|\right). \tag{192}$$

Particularly for $i_0 = 0$ we get for the voltage phase which will make the transient vanish:

$$\psi = \theta \pm \frac{\pi}{2} \tag{192a}$$

which agrees with the above reasoning.

When the transient vanishes, the resulting current is the steady state from the beginning. There is no period of adjustment — commonly called the transient period — because there is no discrepancy to adjust.

The complete picture corresponding to the situation for which Figs. 31 and 32 are drawn is shown in Fig. 33 for the condition $i_0 = 0$.

The usual case in practice is that:

$$L\omega \gg R \tag{193}$$

i.e., the inductive reactance is large compared to the ohmic resistance of the circuit. For this case:

$$\theta \backsim \frac{\pi}{2}$$

so that the steady-state current lags the voltage by practically $\frac{\pi}{2}$ radians or 90°. If the initial current in the circuit is zero, then the condition for the non-appearance of a transient is from (192*a*):

$$\psi = \frac{\pi}{2} \pm \frac{\pi}{2} = \pi,\ 0. \tag{194}$$

Hence in this case, if we throw the switch at the instant the voltage is passing through either its positive or negative maximum,

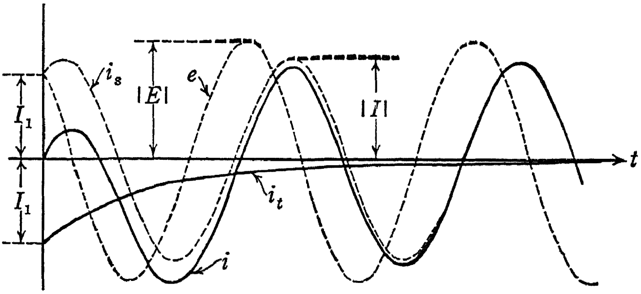


FIG. 33

then the final steady-state current will set in immediately. This is so because the steady-state current is passing through zero at this instant anyway.

Before we leave this case we wish to show how the steady state would be obtained, if we used trigonometric functions for e, and i instead of the complex function. Then eq. (171) would be written:

$$L\frac{di}{dt} + Ri = |E| \cos(\omega t + \psi). \tag{171a}$$

In order to solve we assume:

$$i_s = |I| \cos(\omega t + \phi) \tag{172a}$$

which should be an integral of (171*a*). Substituting this assumption into (171*a*), we get:

$$[-L\omega \sin(\omega t + \phi) + R\cos(\omega t + \phi)]\,|I| = |E| \cos(\omega t + \psi). \tag{173a}$$

From this equation we must now determine $|I|$ and ϕ in terms of $|E|$ and ψ. Note how much more involved this is than the corresponding complex relation (173). The reason for this is twofold. First, (173*a*) involves sines and cosines, whereas (173) involves the same exponential factor throughout which may simply be cancelled out. Second, (173*a*) involves phase angles ϕ and ψ explicitly, which further complicates matters, whereas in (173) the phase angles are concealed, and cause no trouble whatsoever. In order to proceed with (173*a*) we must expand the sines and cosines and collect terms as follows:

$$\begin{aligned}[-L\omega(\sin\omega t\cos\phi + \cos\omega t\sin\phi) &+ R(\cos\omega t\cos\phi - \sin\omega t\sin\phi)]|I| \\ &= |E|(\cos\omega t\cos\psi - \sin\omega t\sin\psi)\end{aligned}$$

or:

$$\begin{aligned}[-(L\omega\cos\phi + R\sin\phi)\sin\omega t &+ (-L\omega\sin\phi + R\cos\phi)\cos\omega t]|I| \\ &= |E|\cos\psi\cos\omega t - |E|\sin\psi\sin\omega t.\end{aligned}$$

Since this is to hold for all values of t, the coefficients of the $\sin\omega t$ and $\cos\omega t$ on either side must be equal. Hence:

$$\left.\begin{aligned}(L\omega\cos\phi + R\sin\phi)|I| &= |E|\sin\psi \\ (-L\omega\sin\phi + R\cos\phi)|I| &= |E|\cos\psi.\end{aligned}\right\} \tag{173b}$$

Squaring both sides of these equations and adding, we obtain:

$$(L^2\omega^2 + R^2)|I|^2 = |E|^2$$

from which finally:

$$|I| = \frac{|E|}{\sqrt{R^2 + L^2\omega^2}}.$$

Now, if we multiply the first eq. (173*b*) by $\cos\phi$, the second by $\sin\phi$ and subtract, we get:

$$|I|L\omega = |E|\sin(\psi - \phi). \tag{173c}$$

If we multiply the first by $\sin\phi$ and the second by $\cos\phi$ and add, we have:

$$|I|R = |E|\cos(\psi - \phi). \tag{173d}$$

Finally dividing (173*c*) by (173*d*), we get:

$$\frac{L\omega}{R} = \tan\theta = \tan(\psi - \phi)$$

so that:

$$\theta = \psi - \phi$$

or:

$$\phi = \psi - \theta$$

which is our result (180).

We have done this merely to show how much more cumbersome the trigonometric functions are to handle. It should be remembered that this illustration was made on a very simple problem. For the case of complicated networks it would be next to impossible to arrive at a solution. At all events the labor involved would be so great that all other points of interest would be submerged. The complex notation is therefore the only logical one to use in treating network problems.

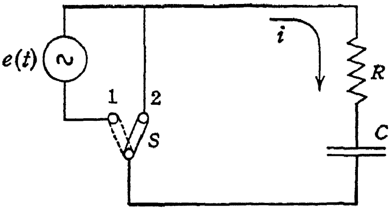


Fig. 34

3. The R, C Circuit. We shall now treat the problem illustrated by Fig. 34. The situation is similar to the one just treated except that a condenser C takes the place of the inductance. $e(t)$ again represents a harmonic voltage and is given by eq. (170). The switch S is thrown from position 2 to 1 at $t = 0$. The differential equation expressing the dynamic equilibrium condition for $t > 0$ is then:

$$Ri + \frac{1}{C}\int i\,dt = Ee^{j\omega t} \tag{195}$$

where the $\Re e$-sign is, of course, dropped for the time being.

The particular integral or steady-state response must be a harmonic function of angular frequency ω. Hence we write for this integral:

$$i_s = Ie^{j\omega t} \tag{196}$$

and substitute this assumption into (195). There results the condition equation:

$$\left(R + \frac{1}{jC\omega}\right)I = E \tag{197}$$

which is easily satisfied by making:

$$I = \frac{E}{R + \dfrac{1}{jC\omega}}. \tag{197a}$$

Hence the steady-state current is given by:

$$i_s = \mathcal{R}_e\left[\frac{Ee^{j\omega t}}{R + \dfrac{1}{jC\omega}}\right]. \tag{198}$$

If we again define:

$$E = |E|\,\underline{|\psi} \tag{199}$$

and:

$$Z = R - \frac{j}{C\omega} = |Z|\,\underline{|\theta} \tag{200}$$

so that:

$$\left.\begin{aligned} |Z| &= \sqrt{R^2 + \frac{1}{C^2\omega^2}} \\ \text{and:}\quad \theta &= -\tan^{-1}\left(\frac{1}{RC\omega}\right) \end{aligned}\right\} \tag{201}$$

then our result (198) may be written:

$$i_s = \mathcal{R}_e\left[\left|\frac{E}{Z}\right|e^{j(\omega t + \psi - \theta)}\right]$$

or:

$$i_s = \left|\frac{E}{Z}\right| \cos\,(\omega t + \psi - \theta). \tag{198a}$$

ψ is the phase angle of the impressed voltage. Hence we see from (198*a*) that the steady-state current lags behind the voltage by the impedance angle θ just as it did in the previous R, L case. But now θ is a negative angle as may be seen from the second equation (201). This means that the current actually leads the voltage by the angle:

$$\tan^{-1}\left(\frac{1}{RC\omega}\right) = \cot^{-1}\,(RC\omega).$$

If the resistance were zero, this angle would be $\frac{\pi}{2}$ radians or 90°. Hence in a pure capacitive circuit the current would lead the voltage by 90°. The effect of the resistance is two-fold. It decreases the magnitude of the current somewhat, and makes the lead angle less than 90°.

It is interesting to note that the d.c. case is contained in this result. We obtain it by making $\omega = 0$. From (200) or (201) we see that this makes the impedance infinite, and hence the steady-state current will vanish as it should.

The steady-state result (198) or its equivalent (198*a*) is illustrated in the Fig. 35. On the left is the vector representation in the complex plane, and on the right is the time plot. The net phase angle for the current is seen from (198*a*) to be:

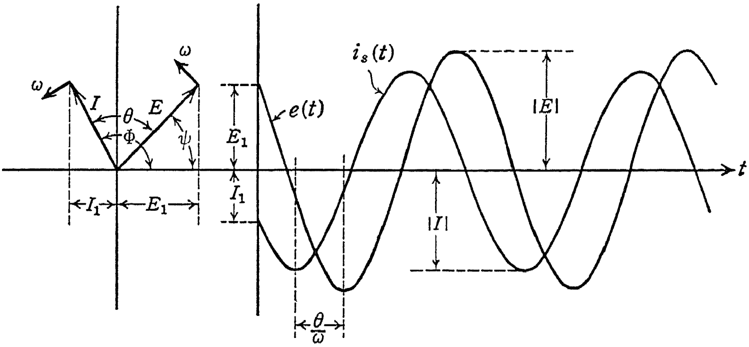


FIG. 35

or:

$$\left.\begin{aligned}\phi &= \psi - \theta \\ \phi &= \psi + \cot^{-1}(RC\omega).\end{aligned}\right\} \tag{202}$$

The complete solution requires the addition of the force-free behavior to the steady state just found. This is given by the solution to the corresponding homogeneous equation which is:

$$Ri + \frac{1}{C}\int i\,dt = 0. \tag{203}$$

But this is, of course, the same as eq. (74), which we already treated in the case of constant excitation. Its solution is given by eq. (78), which we repeat here for convenience:

$$i_t = Ae^{-\frac{t}{RC}} \tag{204}$$

where $-\frac{1}{RC}$ is the natural mode for this system. The complete solution for this case therefore becomes the sum of (198) and (204), thus:

$$i = \mathcal{R}_e\left[\frac{Ee^{j\omega t}}{R + \frac{1}{jC\omega}}\right] + Ae^{-\frac{t}{RC}}. \tag{205}$$

If we use the alternative form (198*a*) for the steady state, this becomes:

$$i = \left|\frac{E}{Z}\right| \cos(\omega t + \psi - \theta) + Ae^{-\frac{t}{RC}}. \tag{205a}$$

This may be put in still simpler form if we note that:

$$\left|\frac{E}{Z}\right| = |I|$$

and write the current phase ϕ in place of $\psi - \theta$, thus:

$$i = |I| \cos(\omega t + \phi) + Ae^{-\frac{t}{RC}}. \tag{205b}$$

The next step is to evaluate the integration constant A. Here we note that we are again treating an impulsive case, i.e., a circuit having no inductance. Hence the initial current is not specified. We have instead to specify the charge in the condenser at $t = 0$, thus:

$$\left.\begin{array}{l}\text{For } t = 0 \\ \text{condenser charge} = \left[\int i\,dt\right]_{t=0} = Q_0.\end{array}\right\} \tag{206}$$

This condition cannot be substituted directly into the complete solution (205). We must first determine the initial current. This we do by going back to the original differential equation (195), and considering it for $t = 0$. This gives us:

$$R(i)_{t=0} + \frac{Q_0}{C} = \mathcal{R}_e[E] \tag{207}$$

from which:

$$(i)_{t=0} = i_0 = \mathcal{R}_e\left[\frac{E}{R}\right] - \frac{Q_0}{RC}. \tag{208}$$

This may be substituted into any of the final forms (205), (205*a*) or (205*b*) for the evaluation of A. Using eq. (205) with a slight modification, we have:

$$\mathcal{R}_e\left[\frac{E}{R}\right] - \frac{Q_0}{RC} = \mathcal{R}_e\left[\frac{E}{Z}\right] + A$$

so that:

$$A = \mathcal{R}_e\left[\frac{E}{R}\right] - \mathcal{R}_e\left[\frac{E}{Z}\right] - \frac{Q_0}{RC}. \tag{209}$$

It is interesting to study the structure of this transient current amplitude. The first term represents the instantaneous value of the impressed voltage at the switching instant divided by the resistance of the circuit. This is the component of current which is immediately established in an R, C circuit when any voltage is impressed; it neglects the presence of the condenser. The last term represents the transient amplitude due to the discharge of the condenser. For positive Q_0 it opposes the first term as indicated. The second term is due to the discrepancy between the steady-state current just before and just after the switching instant. It may be positive or negative depending upon the voltage phase at $t = 0$. The interesting point about eq. (209) is the fact that the resultant amplitude A is simply a linear superposition of these individual effects, i.e., each effect acts as though the others were not present. We call this characteristic of the circuit **linear.** Any network which is made up of constant circuit parameters, i.e., resistances, inductances, and capacitances, behaves linearly in that the net effect of two forces acting in conjunction is equal to the sum of the individual effects. We shall call attention to this important network characteristic again.

Before substituting A of eq. (209) back into the complete solution, we wish to throw it into a more compact form. We did not do this sooner because we wished to call attention to the superposition property above. Eq. (209) may be written:

$$A = \mathcal{R}_e\left[\frac{E(Z - R)}{RZ}\right] - \frac{Q_0}{RC}.$$

Recalling eq. (200) this becomes:

$$A = \mathcal{R}_e\left[\frac{E}{jRC\omega Z}\right] - \frac{Q_0}{RC}$$

or using (199) and (202) also, we have:

$$A = \left|\frac{E}{Z}\right| \frac{\cos\left(\phi - \frac{\pi}{2}\right)}{RC\omega} - \frac{Q_0}{RC}$$

which is equivalent to:

$$A = \frac{|I| \sin \phi - Q_0\omega}{RC\omega}. \tag{209a}$$

If we substitute this into (205*b*), we have the complete solution in final form:

$$i = |I| \cos(\omega t + \phi) + \frac{|I| \sin \phi - Q_0\omega}{RC\omega} \cdot e^{-\frac{t}{RC}}. \tag{210}$$

It is interesting to study this result with the object of determining the conditions under which the transient will be absent. Evidently this will be the case if:

$$|I| \sin \phi = Q_0\omega. \tag{211}$$

If ϕ is fixed, Q_0 must have a definite value in order to suppress the transient. If Q_0 is fixed, ϕ must be properly determined, which requires a proper choice of the switching instant. To illustrate this, suppose the switch is thrown at the instant the generator voltage is passing through its positive maximum. Then:

$$\psi = 0$$

and:

$$\phi = -\theta.$$

Consequently Q_0 must be:

$$Q_0 = \frac{-|I| \sin \theta}{\omega}.$$

If the capacitive reactance is large compared to the resistance, then $\theta \cong -\frac{\pi}{2}$, and:

$$Q_0 \cong \frac{|I|}{\omega} = \left|\frac{E}{Z}\right| \cdot \frac{1}{\omega} \cong |E|C$$

which is the charge necessary to balance the impressed voltage at the initial instant.

If the switch is thrown at the instant the voltage is zero and increasing positively, then:

$$\psi = -\frac{\pi}{2}$$

and:

$$\phi = -\left(\frac{\pi}{2} + \theta\right).$$

Consequently Q_0 must be:

$$Q_0 = \frac{-|I| \cos \theta}{\omega}.$$

For a highly capacitive circuit this is approximately:

$$Q_0 \cong 0$$

which checks with what we should expect.

Suppose finally that Q_0 is given, say Q_1, and it is desired to determine that switching instant which will suppress the transient. Then we have from (211):

$$\sin \phi = \sin (\psi - \theta) = \frac{Q_1\omega}{|I|}$$

or:

$$\psi = \theta + \sin^{-1}\left(\frac{Q_1\omega}{|I|}\right).$$

For a highly capacitive case:

$$|I| \cong |E|C\omega$$

so that this becomes:

$$\psi \cong \sin^{-1}\left(\frac{Q_1}{|E|C}\right) - \frac{\pi}{2}.$$

This equation has real solutions only when:

$$\left|\frac{Q_1}{EC}\right| \leqq 1$$

so that the transient cannot be suppressed by the proper choice of a switching instant unless the charge in the condenser is such that the above inequality is satisfied.

Incidentally a few words should be said about the algebraic sign of the condenser charge. Since the integral in the original differential equation (195) represents the condenser charge, the latter must be defined as positive when it gives rise to a positive counter-voltage, i.e., a voltage acting opposite to the assumed positive direction of the current as indicated by the arrow in Fig. 34. This arrow also indicates the positive direction for the impressed voltage.

There is another form in which we may put the solution to this problem as given by eq. (210), which throws additional light upon the situation. We evaluate the integration constant A directly from (205b) by considering this equation for $t = 0$, thus:

$$i_0 = |I| \cos \phi + A \tag{212}$$

where i_0 is given by (208). But:

$$|I| \cos \phi = I_1$$

is the steady-state current at $t = 0$, so that:

$$A = i_0 - I_1$$

and the final form for the solution is:

$$i = |I| \cos (\omega t + \phi) + (i_0 - I_1)e^{-\frac{t}{RC}} \tag{210a}$$

instead of (210). We wish now to compare this result with that for the R, L circuit given by eq. (189a). If we do this, we see that the R, C case is essentially like that for the R, L circuit. $|I|$ is the magnitude of the steady-state current in both cases. ϕ is the current phase with respect to the reference coördinate. i_0 is the initial current in both cases, and $i_0 - I_1$ is the initial discrepancy which determines the transient amplitude. The latter will vanish whenever this initial discrepancy vanishes. The relations for I, ϕ, and i_0 are, of course, different in the two cases. Also the expressions for the natural modes of decay are different for the two cases. But from the standpoint of form, or outward appearance, the two are exactly alike. They consist of a harmonic steady state with an exponential decay superposed. Inwardly the essential difference between the cases is that the R, C circuit is impulsive, while the R, L one is not. This means that the i_0 for the R, L case will be zero if the circuit is initially at rest, whereas i_0 for the R, C circuit will not be zero for initial rest conditions ($Q_0 = 0$) unless the instantaneous voltage is initially zero.

A plot of the complete solution (210) for the general case, or its equivalent (210a), is shown in Fig. 36 for an arbitrary assumed value of i_0. A comparison of this figure with Fig. 33 for the R, L case will reveal what we have been saying above, namely that the two cases are alike in form except that the R, C case is impulsive as evidenced by the discontinuity in the current at $t = 0$.

It is interesting to determine the condenser voltage for this case. This may be done in various ways. One is to integrate the current expression with respect to time. A simpler way is to obtain it from the original differential eq. (195) directly. The integral term is the condenser voltage. Hence, if we solve for this term, we have:

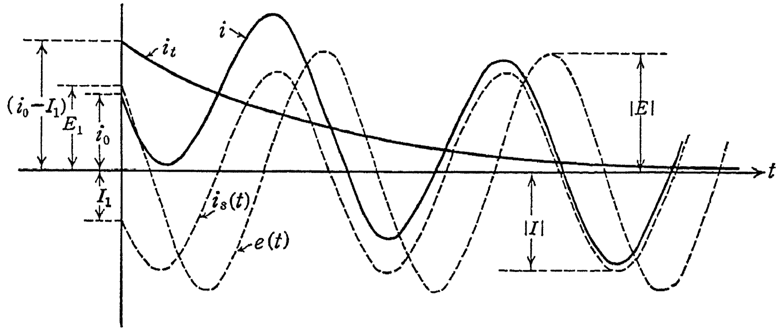


FIG. 36

$$e_c = \mathcal{R}e[Ee^{j\omega t}] - Ri. \tag{213}$$

Substituting (210) into this expression and simplifying, we get:

$$e_c = \frac{|I|}{C\omega} \sin(\omega t + \phi) + \left(\frac{Q_0\omega - |I| \sin\phi}{C\omega}\right) e^{-\frac{t}{RC}}. \tag{213a}$$

We see that the steady-state term has a magnitude equal to the current magnitude times the capacitive reactance, and that it lags the current by $\frac{\pi}{2}$ radians or 90°. The transient amplitude represents the discrepancy between the available condenser voltage at $t = 0$ and that demanded by the steady state. Again the transient is a shock absorber for this discrepancy. If the discrepancy is zero, i.e., if:

$$|I| \sin\phi = Q_0\omega$$

then the condenser voltage shows no transient. Note that this is the condition (211) for the suppression of the transient in the current. Hence, if the transient is suppressed in the current, it is also suppressed in the condenser voltage.

Finally Fig. 37 illustrates the condenser voltage as a function of time according to eq. (213a). This plot corresponds to that in

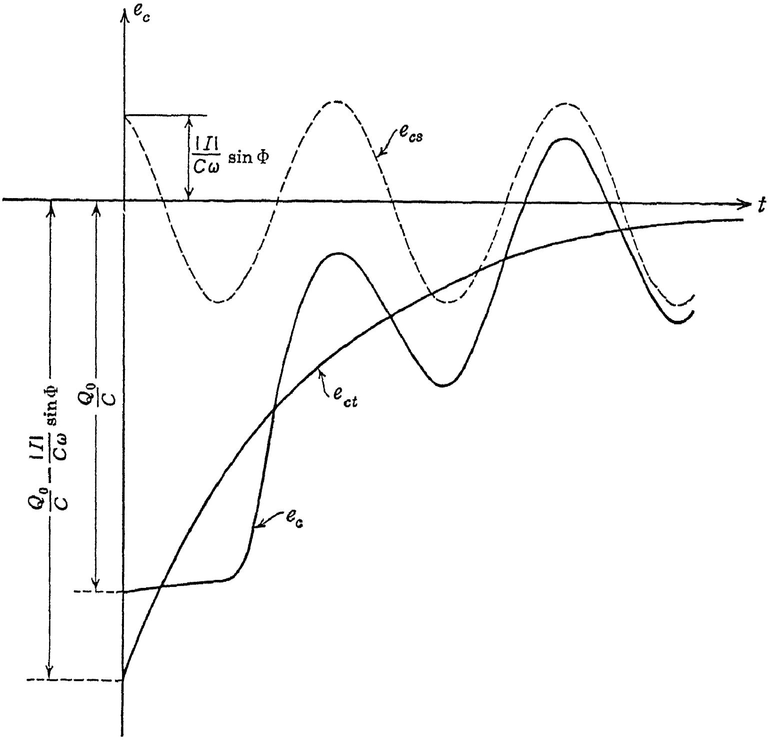


FIG. 37

Fig. 36 for the current. Note that the condenser voltage is continuous throughout the switching instant. Its first derivative is discontinuous at this point which reflects the corresponding current discontinuity. The student should further study Fig. 37 to satisfy himself that it is the integral of Fig. 36, as, of course, it must be.

4. The R, L, C Circuit. We come now to the most interesting case of the single mesh network. All three kinds of elements or parameters are present, and the impressed force is harmonic. The chief interest lies in the variety of possible combinations of circuit and initial conditions which this system displays. The figure 38 gives the circuit and switching arrangement similar to that used before.

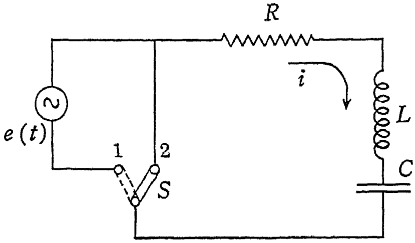


FIG. 38

The equation of dynamic equilibrium for this case is:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = Ee^{j\omega t} \tag{214}$$

where, as usual, the $\Re_e$-operator is dropped on the right-hand side. The assumption for the steady-state current takes the same form as in all network problems with harmonic impressed forces, namely:

$$i_s = Ie^{j\omega t} \tag{215}$$

where the $\Re_e$-operator is also dropped for the sake of convenience. Substitution into (214) gives:

$$\left(Lj\omega + R + \frac{1}{Cj\omega}\right)Ie^{j\omega t} = Ee^{j\omega t}$$

from which:

$$I = \frac{E}{R + j\left(L\omega - \dfrac{1}{C\omega}\right)}. \tag{216}$$

Equation (215) together with (216) represent the steady-state solution. The complex denominator in (216) is the steady-state impedance. The imaginary portion is usually called the reactance, and is often split into two portions: the inductive and the capacitive reactances. It is quite evident that they have opposite algebraic signs. This is due to the fact that the inductance term involves a derivative while the capacitance term involves an integral. The former introduces a j while the latter introduces a $1/j$ or $-j$.

This difference in sign in the reactance part of the impedance is what gives rise to the resonance phenomenon in this circuit. Namely, at that frequency for which the reactance vanishes, the impedance is a minimum, and hence the steady-state current a maximum. Later we shall discover a more significant meaning for this so-called resonance phenomenon whereby we will be able to recognize its general importance in connection with the behavior of physical systems.

For convenience we define:

$$\left.\begin{aligned} Z &= R + j\left(L\omega - \frac{1}{C\omega}\right) \\ Z &= R + jX \end{aligned}\right\} \tag{217}$$

$$\left.\begin{aligned} |Z| &= \sqrt{R^2 + X^2} \\ Z &= |Z|\,\underline{|\theta} \\ \theta &= \tan^{-1}\left(\frac{X}{R}\right) \end{aligned}\right\} \tag{218}$$

and as before:

$$E = |E|\,\underline{|\psi}. \tag{219}$$

Then the steady-state current may be written:

$$i_s = \mathcal{R}_e\left[\frac{E}{Z}e^{j\omega t}\right] \tag{220}$$

or:

$$i_s = \left|\frac{E}{Z}\right| \cos\,(\omega t + \psi - \theta). \tag{220a}$$

If we further define, similarly to what we have done previously:

$$\left.\begin{aligned} I &= |I|\,\underline{|\phi} \\ |I| &= \left|\frac{E}{Z}\right| \\ \phi &= \psi - \theta \end{aligned}\right\} \tag{221}$$

then the steady state may also be written:

$$i_s = |I| \cos\,(\omega t + \phi). \tag{220b}$$

In the R, L case we saw that the steady-state current lagged behind the impressed voltage. In the R, C case it led the voltage. Now it may lead or lag because θ (the lag angle) may be positive or negative according to whether the net reactance X is positive or negative respectively.

If:

$$L\omega > \frac{1}{C\omega}$$

X will be positive, and the steady-state current will lag. We then say that the circuit is preponderantly inductive. If on the other hand:

$$L\omega < \frac{1}{C\omega}$$

then X will be negative, θ will be negative, and the current will lead. We say that such a circuit is preponderantly capacitive. In the resonant case:

$$L\omega = \frac{1}{C\omega}$$

X and θ are, of course, zero, and the circuit acts like a resistance alone so far as the steady state is concerned.

We pass now to the determination of the transient portion of the complete solution. The force-free equilibrium condition is:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0. \tag{222}$$

This is, of course, identical with the force-free equilibrium condition for the case of the R, L, C circuit with constant excitation treated in Chapter II. Hence its solution is identical with that found there. For convenience we will repeat the essential steps here. The assumed solution of (222) is as usual for the natural behavior of a network:

$$i_t = Ae^{pt}. \tag{223}$$

Substitution into (222) leads to the condition:

$$\left(Lp + R + \frac{1}{Cp}\right)Ae^{pt} = 0$$

which, for the existence of non-trivial solutions, leads to the further condition:

$$p^2 + \frac{R}{L}p + \frac{1}{LC} = 0 \tag{224}$$

which is the determinantal equation for the system. Its roots are the natural modes. If we define:

$$\left.\begin{aligned} \alpha &= \frac{R}{2L} = \text{damping constant} \\ g_0 &= \frac{1}{\sqrt{LC}} = \text{undamped angular velocity} \\ g &= \sqrt{g_0^2 - \alpha^2} = \text{damped angular velocity} \end{aligned}\right\} \tag{225}$$

then the roots of (224) are given by:

$$\left.\begin{aligned} p_1 &= -\alpha + jg \\ p_2 &= -\alpha - jg \end{aligned}\right\} \tag{226}$$

provided:

$$g_0 > \alpha. \tag{227}$$

The condition (227) is necessary in order that the square root expression for g be real so that the roots (226) will be a pair of conjugate complexes. It will be remembered that this is the oscillatory case. The aperiodic and critically damped cases, defined by $g_0 < \alpha$ and $g_0 = \alpha$ respectively, will not be taken up here. They are very similar in nature to the R, L and R, C cases just discussed. The student may work these out for himself as practice problems. The case of most interest and importance is the oscillatory one. It is the one which occurs most frequently in communication work.

Thus we proceed with the information that the assumption (223) will be valid and useful provided either of the roots (226) are used for p. Hence we see that the complete transient solution is given by:

$$i_t = A_1 e^{p_1 t} + A_2 e^{p_2 t}. \tag{228}$$

It remains now to add this transient portion to the steady state in order to form the complete solution, and then proceed to evaluate the integration constants A_1 and A_2 from specified initial conditions. Before doing this, however, we wish to call attention to a simplification which can be made at this point. We recall that in this same case with constant excitation, A_1 and A_2 turned out to be a pair of conjugate complex numbers. We pointed out there that a pair of constants which are associated with exponentials involving a pair of conjugate modes, always were conjugate complexes themselves. Hence (228) may be written:

$$i_t = 2\,\mathcal{R}_e[A_1 e^{p_1 t}]$$

which involves only one constant and one natural mode. Since there is only one of each, let us drop the subscripts and simply write:

$$i_t = \mathcal{R}_e[Be^{pt}] \tag{228a}$$

where the single mode is:

$$p = -\alpha + jg \tag{226a}$$

and B is a new constant.

Thus the complete solution becomes the sum of (220) and (228a), which is:

$$i = \mathcal{R}_e(Ie^{j\omega t} + Be^{pt}) \tag{229}$$

where I is given by (216). This is a very compact way of writing the complete solution.

For the evaluation of B let us suppose that we have the following initial conditions specified:

$$\left.\begin{array}{l}\text{For } t = 0\\ \text{condenser charge } = Q_0\\ \text{current } = i_0.\end{array}\right\} \tag{230}$$

It should be noted that two initial conditions may be specified, because the constant B is complex and therefore really splits into two constants, thus:

$$B = B_1 + jB_2. \tag{231}$$

The initial charge cannot be substituted into (229) since this is an equation in terms of current. However, (229) may be differentiated without any trouble. Hence, if we knew what the initial time derivative of the current were, then we could use this in (229). This initial time derivative is easily found by going back to the original differential equation (214) and considering it for $t = 0$. With the information given by (230) this becomes:

$$L\left(\frac{di}{dt}\right)_{t=0} + Ri_0 + \frac{Q_0}{C} = \mathcal{R}_e[E]. \tag{232}$$

This equation may now be solved for the initial derivative of the current.

Letting:

$$E = E_1 + jE_2 \tag{233}$$

we have:

$$\left(\frac{di}{dt}\right)_{t=0} = \frac{E_1}{L} - \frac{Ri_0}{L} - \frac{Q_0}{LC}. \tag{234}$$

Using this and the last relation (230) in (229), we have:

$$i_0 = \mathcal{R}_e(I + B) \tag{235}$$

and:

$$\frac{E_1}{L} - \frac{Ri_0}{L} - \frac{Q_0}{LC} = \mathcal{R}_e(Ij\omega + Bp). \tag{236}$$

If we let:

$$I = I_1 + jI_2 \tag{237}$$

we get:

$$B_1 = i_0 - I_1 \tag{238}$$

and:

$$B_2 = I_1\frac{\alpha}{g} - I_2\frac{\omega}{g} - \frac{E_1}{Lg} + i_0\frac{\alpha}{g} + Q_0\frac{g_0^2}{g}. \tag{239}$$

The last equations (238) and (239) together with (229) represent the complete solution to this problem under any assumed initial conditions. In order to simplify the discussion of this result, we will assume that the system is initially at rest so that:

$$i_0 = Q_0 = 0. \tag{240}$$

Furthermore we shall assume that the circuit is highly oscillatory, i.e., that:

$$\alpha \ll g \qquad g \cong g_0. \tag{241}$$

Then the relations (238) and (239) for the real and imaginary portions of B become:

$$B_1 = -I_1 \tag{238a}$$

and:

$$B_2 = -I_2\frac{\omega}{g} - \frac{E_1}{Lg}. \tag{239a}$$

These conditions are those which are most commonly found in communication work.

We are now going to discuss the above general result under various special conditions which may be summarized as follows:

I. Voltage phase $\psi = 0$.

$$\begin{aligned}&(a)\ \omega = g.\\&(b)\ \omega \backsimeq g.\\&(c)\ \omega \gg g.\\&(d)\ \omega \ll g.\end{aligned}$$

II. Voltage phase $\psi = -\frac{\pi}{2}$.

$$\begin{aligned}&(a)\ \omega = g.\\&(b)\ \omega \backsimeq g.\\&(c)\ \omega \gg g.\\&(d)\ \omega \ll g.\end{aligned}$$

There are, of course, innumerable other intermediate cases, but the above summary will give the reader a good idea of the variety of results obtainable with this simple circuit.

We consider first case I*a*:

$$\psi = 0 \qquad \omega = g.$$

Here we have:

$$E_1 = E \qquad I = \frac{E}{R} = I_1 \qquad I_2 = 0$$

so that:

$$B_1 = -\frac{E}{R} \qquad B_2 = -\frac{E}{Lg}.$$

Hence the complete solution (229) becomes:

$$i = \mathcal{R}_e\left[\frac{E}{R}e^{j\omega t} - \frac{E}{R}e^{pt} - j\frac{E}{Lg}e^{pt}\right]$$

or:

$$i = \frac{E}{R}(1 - e^{-\alpha t})\cos\omega t + \frac{E}{L\omega}e^{-\alpha t}\sin\omega t. \tag{242}$$

Notice that the second term in this expression represents the result of impressing a constant voltage E upon an R, L, C circuit. It is a damped sinusoid, with an amplitude equal to the voltage divided by the inductive reactance. We could write it as the voltage divided by the capacitive reactance just as well since this is the resonance case. The first term has a much larger amplitude and consists of a cosine function inside of an exponential envelope. Since we have assumed the circuit to be highly oscillatory, the second term in (242) is negligible compared to the first so that we can write:

$$i = \frac{E}{R}(1 - e^{-\alpha t}) \cos \omega t. \tag{242a}$$

A plot of this result is shown in Fig. 39. It is simply an exponential build-up to the asymptotic value E/R.

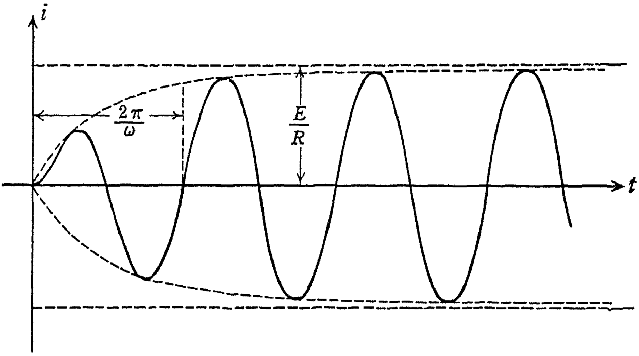


FIG. 39

We come now to case Ib:

$$\psi = 0 \qquad \omega \cong g.$$

This is the so-called near-resonance case. So far as the evaluation of the integration constant is concerned, we arrive at the same result as in case Ia above, namely:

$$B_1 = -\frac{E}{R} \qquad B_2 = -\frac{E}{Lg}.$$

Again we will neglect B_2 compared to B_1, and write the final result as:

$$i = \Re\left[\frac{E}{R}(e^{j\omega t} - e^{-\alpha t} \cdot e^{jgt})\right]. \tag{243}$$

The difference between this and the previous case is that now we cannot factor out the exponential $e^{j\omega t}$ because $\omega \neq g$. Taking the real portion of the above we have:

$$i = \frac{E}{R}(\cos \omega t - e^{-\alpha t} \cos gt). \tag{243a}$$

In order to interpret this result let us neglect the damping factor $e^{-\alpha t}$ temporarily. Then by means of trigonometry we have:

$$
\begin{aligned}
i &= \frac{E}{R}(\cos \omega t - \cos gt) \\
&= \frac{2\,E}{R} \cdot \sin\left(\frac{g-\omega}{2}\right)t \cdot \sin\left(\frac{g+\omega}{2}\right)t. \qquad (244)
\end{aligned}
$$

This is the product of two harmonic functions. The first has a very long period, while the second has an angular frequency very nearly equal to either ω or q. Hence the first factor acts as an envelope for the second. This is plotted in Fig. 40. Note that the amplitude of the envelope is twice the steady-state amplitude.

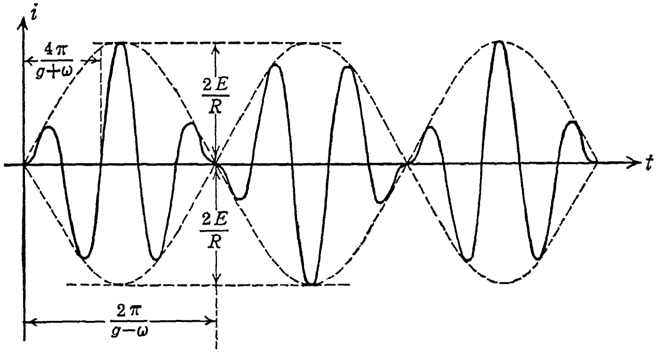


FIG. 40

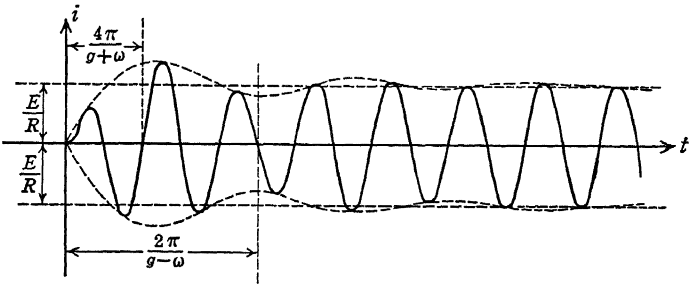


FIG. 40*a*

Due to the damping factor, the second term in (243*a*) eventually goes to zero so that the beats shown in Fig. 40 cease. The actual performance of the circuit is shown in Fig. 40*a*. If we compare this with Fig. 39 for $\omega = q$, we see that now the circuit builds up too fast and overshoots its steady-state amplitude, reaching almost twice normal value at the maximum of the first beat. On the return from this value it overshoots the steady-state value again, and reaches a minimum value which will be determined by the amount of damping up to that point. This beating about the steady-state value continues until the second term in (243*a*) is damped out.

Note that the beat frequency is determined by the difference between the natural and the impressed frequencies. As these two approach each other, the beat period lengthens more and more until the behavior of Fig. 40*a* finally goes over into that of Fig. 39.

The next case in order is I*c*:

$$\psi = 0 \qquad \omega \gg g$$

i.e., the impressed frequency is high compared to the natural frequency of the circuit. This case is seldom met in practice, but we shall treat it here nevertheless. First let us see what the above condition means in more familiar terms. If:

$$\omega \gg g$$

then obviously:

$$\omega \gg g_0$$

so that:

$$\omega^2 \gg g_0^2 = \frac{1}{LC}$$

or:

$$L\omega \gg \frac{1}{C\omega}.$$

Thus we see that if the impressed frequency is large compared to the natural frequency of the circuit, it means that the inductive reactance is large compared to the capacitive reactance. The circuit acts very much as though it were altogether inductive, since the resistance plays only a minor rôle by comparison. Hence we have:

$$E_1 = E \qquad I \cong \frac{E}{jL\omega} \qquad I_1 \cong 0 \qquad I_2 \cong -\frac{E}{L\omega}.$$

The voltage vector is purely real. The current vector is substantially negative imaginary, i.e., ϕ is $-\frac{\pi}{2}$. Using these relations in equations (238a) and (239a) we get:

$$B_1 = 0 \quad \text{and} \quad B_2 = \frac{E}{L\omega} \cdot \frac{\omega}{g} - \frac{E}{Lg} = 0.$$

Hence the transient is suppressed altogether. The complete solution is given by the steady state alone:

$$i = |I| \cos\left(\omega t - \frac{\pi}{2}\right) = \frac{E}{L\omega} \sin \omega t. \tag{245}$$

Fig. 41 shows the voltage and current for this case. The result is as though the circuit contained nothing but an inductance. This is so because the inductance is so important compared to the rest of the parameters under the above assumed conditions.

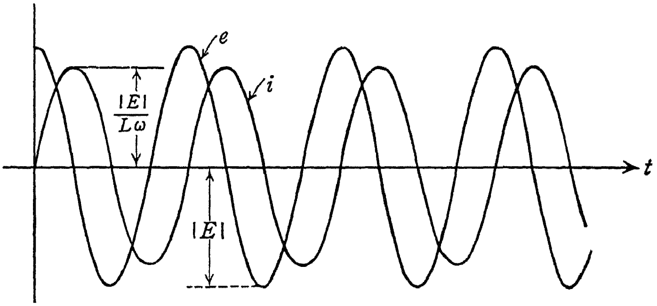


FIG. 41

We now turn to case Id:

$$\psi = 0 \qquad \omega \ll g.$$

By means of the same reasoning as above, we see that this is equivalent to:

$$L\omega \ll \frac{1}{C\omega}$$

i.e., the capacitive reactance is now the predominant factor. Let us see what follows in this case. We have:

$$E_1 = E \qquad I = EjC\omega \qquad I_1 \cong 0 \qquad I_2 \cong EC\omega.$$

Again the current vector is substantially purely imaginary, but leading the voltage by $\frac{\pi}{2}$ this time, i.e., $\phi = \frac{\pi}{2}$. Substituting these results into equations (238*a*) and (239*a*) we get:

$$B_1 = 0 \qquad B_2 = -ECω \cdot \frac{ω}{g} - \frac{E}{Lg}.$$

But:

$$\frac{E}{Lg} = \frac{ECω}{LCωg} \cong ECω \cdot \frac{g}{ω}$$

so that:

$$B_2 = -ECω\left(\frac{ω}{g} + \frac{g}{ω}\right) \cong -ECω \cdot \frac{g}{ω}$$

because $g \gg ω$. This is a surprising result. It is equivalent to:

$$B = -I \cdot \frac{g}{ω} \tag{246}$$

i.e., the transient current vector is opposite in phase to the steady-state current vector, but much larger in magnitude. The ratio is $g/ω$ which may be ten times or more in many cases. This means that the transient disturbance will be exceptionally large and may last for an appreciably long time unless the damping is high. The vector diagram for $t = 0$ is given by Fig. 42. The voltage vector is in the reference position on account of $\psi = 0$. The steady-state current vector leads by practically $\frac{\pi}{2}$ radians. The transient current vector is opposite and much longer. As time increases from zero all vectors rotate in the positive direction. E and I rotate at $ω$ radians per second, and B rotates at g radians per second, which is much faster — as much faster as B is longer than I, so that it may make ten revolutions or more to one revolution of I or E. As it rotates it decays due to $α$ so that eventually it will disappear altogether. Nevertheless there will be times when B and I act in conjunction to form large projections upon the horizontal axis, and thus give rise to a large net instantaneous current. In effect we have a slow steady-state oscillation of small amplitude with a high frequency damped oscillation of very

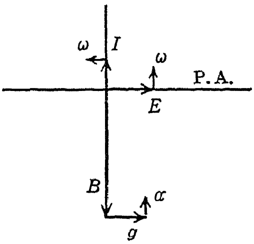


FIG. 42

large amplitude superposed. Analytically the result is obtained by substituting (246) into (229), thus:

$$i = \Re_e\left[I\left(e^{j\omega t} - \frac{g}{\omega}e^{-\alpha t}e^{jgt}\right)\right]$$

or:

$$i = -\,EC\omega\left[\sin \omega t - \frac{g}{\omega}e^{-\alpha t}\sin gt\right]. \qquad (247)$$

A plot of this result is shown in Fig. 43. The high-frequency oscillations at the start almost obliterate the presence of the steady

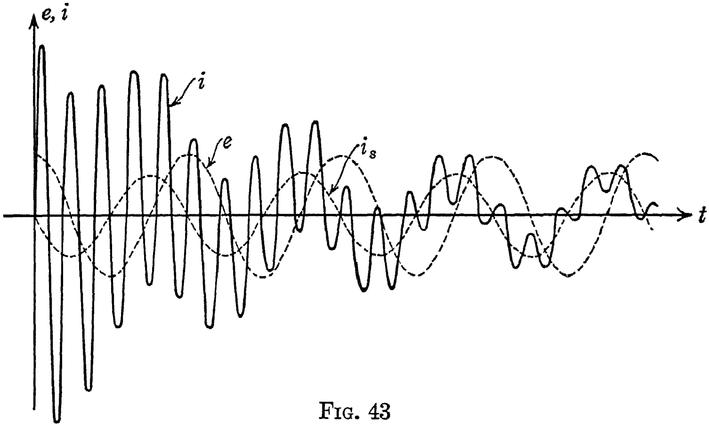


FIG. 43

state altogether. The latter gradually emerges out of the initial chaos and eventually becomes recognizable.

This result is rather surprising, especially after considering the highly inductive case I*c*, where the result was rather unexciting because it was substantially like that for an inductance alone. The reader really expects at first that case I*d* will turn out to be very much like that for a condenser alone. The reason for the appearance of such a violent transient in the latter case may be seen from the following reasoning. In both cases I*c* and I*d* the switch is thrown at the instant the voltage passes through its maximum. If we recall the problem of suddenly applying a constant voltage to an R, L, C circuit, treated in Chapter II, we see that this sudden application of voltage at the first instant creates a vector of magnitude:

$$\frac{E}{Lg}$$

which immediately starts to rotate and decay according to the natural mode of the circuit, thus producing a damped sinusoid of the above amplitude and natural frequency. This accounts for the second term in eq. (239*a*) for B_2. In order to give the first term a physical meaning, we must get a better idea of the vector representation of a harmonic function. In order to do this, consider for a moment the term:

$$\mathcal{R}_e[Be^{jgt}]$$

with:

$$B = B_1 + jB_2.$$

The initial magnitude of this function is:

$$\mathcal{R}_e[B] = B_1.$$

The initial rate of change is:

$$\mathcal{R}_e[jgB] = -\ gB_2.$$

This may be realized from the vector diagram of Fig. 44. Instead of having B rotate at g radians per second, we may replace this by the two component vectors B_1 and B_2 rotating with the same angular velocity. The B_1 vector then represents a cosine function, whereas the B_2 vector alone represents a sine function. At the initial instant, the magnitude of the entire function is given by its cosine part alone because the sine part is initially zero. The rate of change, however, is initially determined by the sine part alone because the cosine part has no initial rate of change. Thus B_1 governs the initial magnitude of the entire function, while B_2 — or more accurately B_2g — governs its initial rate of change.

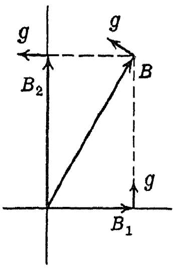


FIG. 44

With this in mind let us reconsider the equations (238*a*) and (239*a*), which express the transient current amplitude in terms of the steady state. Eq. (238*a*) is obviously an equation expressing the condition of continuity of current magnitude. This magnitude before $t = 0$ is zero. Hence it must be zero just after $t = 0$. The initial transient magnitude must therefore be equal and opposite to the initial steady-state magnitude. The second eq. (239*a*) expresses a voltage balance at the initial instant. Since the current and charge are both initially zero, the only place that a voltage can appear is at the inductance. Hence the equation of voltage balance becomes an equation of the balance of initial rates of change of current. The steady-state rate is $I_2\omega$ for the reason explained above. The sudden introduction of a vector of magnitude E_1/Lg rotating at g radians per second produces another initial component of current rate of change equal to E_1/L. The transient must balance both of these so that B_2q must be equal and opposite to them. This gives rise to the eq. (239*a*).

Now in the highly inductive case I*c* the steady-state rate $I_2\omega$ cancels the rate E_1/L because they represent equal vectors in opposite directions. In the highly capacitive case I*d* this cancellation does not take place. In fact the rate $I_2\omega$ is negligible compared to E_1/L. This requires the introduction of a transient component vector B_2 substantially equal to E_1/Lg, and hence very much larger than the steady-state current vector. The student should illustrate these points more fully for himself by the help of vector diagrams.

The above discussion illustrates a very fundamental point, namely that the evaluation of the integration constant — which is the determination of the transient current amplitude — is nothing more than an adjustment of the complete solution so as to meet the conditions of current and voltage balance at the initial instant. At all subsequent instants this balance is taken care of automatically by the form of the solution. In every case except the impulsive case, the voltage balance becomes a balance of initial rates of change.

We have yet to consider the Cases II, *a*, *b*, *c*, *d*. These are the same as the corresponding cases I except that the switch is thrown at the instant the voltage is passing through zero and increasing positively. Consider case II*a*:

$$\psi = -\frac{\pi}{2} \qquad \omega = g.$$

Here:

$$E = -j|E| \qquad E_1 = 0 \qquad I = \frac{E}{R} \qquad I_1 = 0 \qquad I_2 = -\frac{|E|}{R}.$$

Hence by (238*a*) and (239*a*):

$$B_1 = 0 \qquad B_2 = \frac{|E|}{R} \tag{248}$$

so that the complete solution becomes:

$$i = \frac{|E|}{R} \mathcal{R}e[-je^{j\omega t} + je^{-\alpha t}e^{jgt}] \tag{249}$$

or:

$$i = \frac{|E|}{R}(1 - e^{-\alpha t}) \sin \omega t. \tag{249a}$$

This result is practically the same as the corresponding case I*a*, the solution for which is given by eq. (242*a*) and plotted in Fig. 39.

For case II*b* we have:

$$\psi = -\frac{\pi}{2} \qquad \omega \cong g.$$

The result will be the same as before except that the step from (249) to (249*a*) cannot be made. We have instead:

$$i = \frac{|E|}{R}[\sin \omega t - e^{-\alpha t} \sin gt]. \tag{250}$$

This is essentially the same as case I*b* given by eq. (243*a*) and illustrated in Fig. 40*a*.

Next in order is case II*c*:

$$\psi = -\frac{\pi}{2} \qquad \omega \gg g.$$

Hence we have:

$$E = -j|E| \qquad E_1 = 0 \qquad I = \frac{E}{jL\omega} \qquad I_1 = -\frac{|E|}{L\omega} \qquad I_2 = 0$$

so that eqs. (238*a*) and (239*a*) give:

$$B_1 = \frac{|E|}{L\omega} \qquad B_2 = 0. \tag{251}$$

The complete solution for this case is therefore by eq. (229):

$$i = -\frac{|E|}{L\omega}(\cos \omega t - e^{-\alpha t} \cos gt). \tag{252}$$

This is a high-frequency steady state superposed upon a low-frequency decaying transient of equal initial amplitude, as illustrated by Fig. 45. The decaying transient acts as an axis for the steady state to oscillate about. The first maximum of the current is almost twice normal. Notice the considerable difference between this case and the corresponding case I*c* illustrated by Fig. 41. The difference is due merely to the different choice of the switching instant, i.e., the voltage phase ψ.

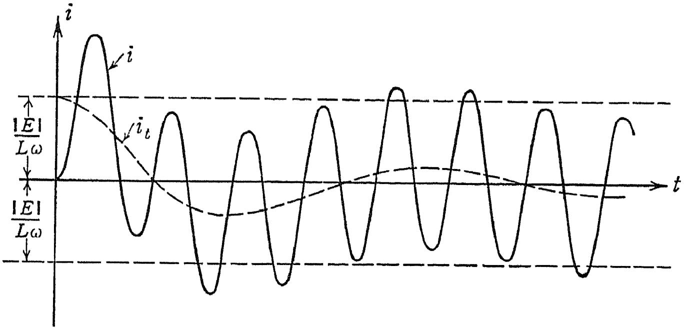


FIG. 45

Finally we consider the case II*d*:

$$\psi = -\frac{\pi}{2} \qquad \omega \ll g.$$

For this condition it follows that:

$$E = -j|E| \qquad E_1 = 0 \qquad I = jEC\omega \qquad I_1 = |E|C\omega \qquad I_2 = 0$$

so that the transient amplitudes become by eqs. (238*a*) and (239*a*):

$$B_1 = -|E|C\omega \qquad B_2 = 0. \tag{253}$$

Hence the complete solution (229) is:

$$i = |E|C\omega \cdot \mathcal{R}_e[e^{j\omega t} - e^{-\alpha t} \cdot e^{jgt}] \tag{254}$$

or:

$$i = |E|C\omega(\cos \omega t - e^{-\alpha t} \cos gt). \tag{254a}$$

This is a high-frequency decaying transient superposed upon a low-frequency steady state of equal initial amplitude as illustrated by Fig. 46. This case is in a way the reciprocal of the preceding one shown in Fig. 45 in that the steady state now acts as an axis for the decaying transient to oscillate about. The first current maximum is again almost double the final steady state. It is interesting to note the considerable difference between this case and the corresponding case I*d* illustrated in Fig. 43. There the result was also a decaying transient oscillating about the steady state as an axis, but the initial amplitude of this transient was many times that of the steady state. In the present case it is just equal to the steady state amplitude. The difference is merely due to the difference in the switching instants or the voltage phases.

In taking up the preceding eight special cases we intended to give the reader a survey of the variety of results that can be obtained with this really simple R, L, C circuit under various conditions of excitation and relative magnitudes of the circuit parameters and the impressed frequency. We have by no means exhausted the situation in this respect. Besides the existence of innumerable intermediate cases between the above, we have not even touched upon anything but initial rest conditions. The available current and condenser charge at $t = 0$ may be chosen in ways to either aggravate or diminish the transient in any of the above cases. This study must be made using the eqs. (238) and (239) for the transient current vector instead of (238a) and (239a). Obviously this presents a host of additional typical cases. Furthermore we have not yet investigated the behavior of the inductance or capacitance voltages in any of the cases.

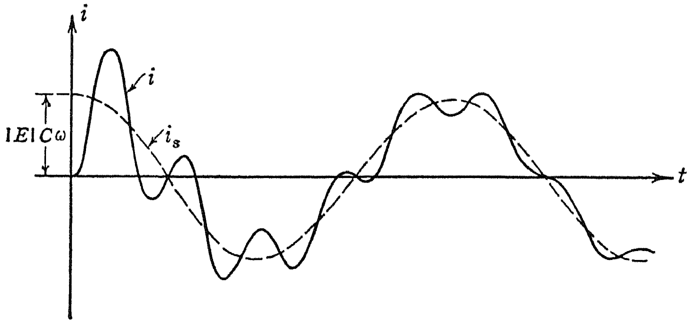


Fig. 46

We prefer, however, not to go any further into these matters here in the text. They may make a fruitful field wherefrom to make up practice problems for further study. We have shown the reader the fundamental methods whereby almost any question concerning this simple circuit may be answered. We will not spoil his practice field by working out everything for him.

5. Further Considerations of the Steady State. Before introducing the subject of this section we should like to summarize what we have covered so far. In the opening chapter we gave the reader an idea of the factors which characterize an electrical network, and of the means whereby a solution to a given situation might be obtained. We spoke of mechanical analogies as aids in visualizing the network performance. The differential equation was introduced as a physical expression of the dynamic equilibrium of impressed and counter-forces. The complete solution, in the form of a superposition of two major parts, was interpreted physically so that the process of obtaining it would appear in the light of a logical succession of steps rather than an arbitrary application of formulae and rules. The heuristic method of approach was introduced as the logical way of attacking the solution to any problem that has a physical background. Integration constants and special circuit conditions were correlated and presented in the light of a process of mutual agreement between arbitrary freedom in the formal solution and a like number of physical specifications. And, among the points of outstanding importance, the complex representation of a harmonic function was introduced.

In the succeeding chapters, up to the present point, these introductory ideas were discussed in some detail with the single mesh network as a vehicle for illustration. In the paragraphs below we are concerned with matters which further aid in correlating some of the ideas already touched upon.

In a great deal of our work with electrical networks we are not so vitally interested in knowing just what the instantaneous value of current is at every instant following some switching operation, with given circuit conditions at the initial instant, as was discussed above in connection with various simple circuit arrangements. Only in relatively few instances does the so-called transient behavior become important, as for example in the case of relay-operation and the like. In such cases the behavior is based upon the transient almost entirely. Time delay devices also are frequently based upon transient behavior. But these problems occur relatively seldom as compared to problems which concern themselves with the steady-state behavior alone.

Take, for example, some link in a radio communication system like the tuned-circuit receiver. Here we wish to know how this system behaves continuously under the influence of a steady-applied driving force. It is true, of course, that the magnitude of the driving force in most cases does not remain constant as we have been assuming in the above illustrations. In the case of reception of electrical impulses due to speech or musical sounds neither the amplitude nor the frequency of the impressed forces remain constant for any length of time. In fact they are changing continuously. By a steady state, on the other hand, is meant a process which continues in the same way forever. Strictly speaking, a mathematical steady state is one which has behaved and will continue to behave forever in the same way that it is at the time it is being considered. Under this definition nothing in nature is a steady state. It is quite obvious then that the engineer must re-define the term steady state in such a way as to obtain a more useful concept.

The term steady state in engineering is always used relatively. Whether or not a given process is considered as a steady state depends upon the nature of the system in which it is taking place. We spoke in some of the foregoing illustrations about what we called the time-constant of a circuit. This time-constant was defined as the reciprocal of the damping factor, which is usually denoted by α. It is that length of time which will make the exponential term $e^{-\alpha t}$ equal to e^{-1} or 0.368. Since this exponential term governs the magnitude of the corresponding transient term, it follows that the time constant is a measure of the length of time that the transient will be important in affecting the character of the net result. After a time which is large compared to the time-constant, the remaining current or voltage will vary substantially in accordance with the steady state alone.

Practically then we do not have to wait forever in order to obtain what is substantially a steady-state performance. In fact for most communication networks the time-constants are extremely small, so that a process that has continued only for a small fraction of a second may already be considered as a steady state. In general, if the process has continued for a time which is large compared to the time-constants involved, then it may be considered as a steady state. This is the engineer's definition of steady state. As pointed out above, it is entirely relative. More than that, it is almost a matter of taste as to what we might consider as "long compared to the time-constant." This is a matter about which no rules can be given. It depends upon too many factors, such as the use to which the particular result is to be put, the accuracy desired, etc. It is akin to the factor of safety used in construction work, and can be estimated only in connection with a specific problem, based upon previous experience.

The student need not worry particularly over this point at present. It will suffice to say that in the light of the above discussion such phenomena as speech and musical sounds are frequently considered as a smooth succession of steady states in communication work. This is certainly a far cry from the mathematical definition of steady state given above. The only justification for proceeding upon this basis is that the results obtained on this assumption are usually sufficiently substantiated by experiment, which is in the end the engineer's final court of appeal. In order to temper the exaggeration somewhat we frequently refer to such phenomena as quasi-steady states. By this we in effect admit that we are treating a situation as a steady state, whereas we really know better, but continue in spite of this knowledge because a rigorous treatment is so involved as to be out of the question. We shall say more about the rigorous treatment in some of the following chapters.

The considerations just made lead us to recognize more in the steady-state solution than we have heretofore pointed out. So far we have considered the steady-state solution merely as a mathematical function of time. Now we see that in communication work we must consider it as a function of frequency as well, for not only must we know how our network reacts to one frequency, but we must know how it will react to any frequency within the region which is being covered in our particular case. We say that we must consider the solution for a continuous **band** of frequencies within which the vital frequencies lie.

With these points in mind, let us return again to the steady-state solution of our R, L, C circuit:

$$i_s = \mathcal{R}_e\left[\frac{Ee^{j\omega t}}{R + j\left(L\omega - \dfrac{1}{C\omega}\right)}\right]. \tag{255}$$

Even this is more specific than necessary. It is understood that the steady-state current is harmonic and has an angular frequency of ω radians per second. Hence our chief interest lies in knowing the magnitude of the oscillation, or more particularly, the root-mean-square amplitude. If we let E and I designate root-mean-square amplitudes, then:

$$\left.\begin{aligned} I &= \frac{E}{Z} \\ \text{with:}\qquad Z &= R + j\left(L\omega - \frac{1}{C\omega}\right). \end{aligned}\right\} \tag{256}$$

Thus we see that this root-mean-square current amplitude depends upon ω, and hence upon frequency. But notice that this functional dependence upon ω is confined to the impedance Z alone. This impedance contains all the information regarding the dependence of I upon ω so long as E remains fixed, which for the time being we shall assume. We call Z not merely an impedance, but an **impedance function.** This property of Z is placed in evidence mathematically by writing $Z(\omega)$ instead of merely Z. If we know its behavior regarding ω, then that for I follows easily.

Let us leave this for a moment to consider what bearing the natural modes of the network might have upon the matter under discussion. Taking the same R, L, C case, let us recall the determinantal equation of the circuit:

$$D(p) = Lp + R + \frac{1}{Cp} = 0 \tag{257}$$

where the symbol $D(p)$ is introduced to stand for the left-hand side of this equation when considered as a function of p. We notice at once that the functions $D(p)$ and $Z(\omega)$ bear a close resemblance. The outward reason for this resemblance is due to the fact that the assumed steady state and transient solutions:

$$Ie^{j\omega t}$$

and:

$$Ae^{pt}$$

respectively, differ (besides in amplitude) only by the appearance of p in the place of $j\omega$. This in fact is the only difference between $Z(\omega)$ and $D(p)$.

The roots of the determinantal eq. (257) we have called the natural modes. They are those values of p for which $D(p)$ will vanish. We have seen, and shall see more thoroughly later, that when they are complex, they always come in conjugate pairs. We also know that one pair together defines only one decrement and one angular frequency, the latter being the imaginary portion common to the pair of conjugate modes.

Returning to the steady-state expression (256) again, suppose we had a generator with a complex frequency, i.e., one whose voltage were given by:

$$e(t) = Ee^{(-\alpha+jg)t}$$

where the complex frequency $-\alpha + jg$ was a root of (257) or a natural mode of the circuit. This, of course, represents a damped voltage. This complex frequency would make $Z(\omega)$ vanish just as it makes $D(p)$ vanish. Assuming that a steady state could be obtained, it would be infinite. Practically, this is out of the question, of course, because such an impressed voltage as the above would vanish with time, and hence the current would vanish also. But we can carry out our supposition anyway if we imagine that the initial amplitude is large enough to start with. Then the steady-state current I will tend toward infinity so long as the voltage does not give out. This is due to the fact that the impressed frequency in this hypothetical case coincides exactly with one of the natural complex frequencies or modes. This would be real resonance — super-resonance.

If instead of this hypothetical case, we consider the usual case but with $\omega = g$, i.e., the impressed angular velocity equal to the natural angular velocity, then $Z(\omega)$ does not vanish, but it becomes pretty small, in fact equal to R alone in the present case. This is the usual condition which we refer to as resonance. Resonance from now on should convey a more significant meaning to us than merely the fact that the impedance function has reached a minimum. It means coincidence between the impressed angular velocity and one of the natural angular velocities — or the imaginary portion of one of the natural modes of the system. This immediately gives us a clearer picture of both the resonance phenomenon and the natural modes. The system is being driven in agreement with — or almost in agreement with — one of its natural tendencies. The resulting amplitude is exceptionally large. We will see later that more complicated networks possess a larger number of natural modes, and hence we see that these systems must possess a correspondingly larger number of resonant frequencies and resonant responses in their meshes.

We return again to the R, L, C circuit and the eqs. (256) in order to discuss these in more detail. In radio reception as well as in other branches of communication work it becomes necessary to study the response of a circuit as a function of frequency, i.e., to study the relations (256) as functions of ω. If we were to plot the magnitude of the current given by the first of these equations against ω, the result would look like the curve of Fig. 47. At the resonance point $\omega = g_0$ the response is exceptionally good. To either side of this point the curve drops off. If we are interested in the so-called selectivity of the circuit, then we must have some idea of the rapidity with which the curve drops off to either side of the resonant point. In fact this characteristic is a direct measure of the selectivity. In a general way we can see from the analytic form of $Z(\omega)$ in this case that the selectivity will be better the smaller the resistance becomes, because this increases the maximum very materially. Let us see, however, whether we can give this matter a more definite mathematical form. We have:

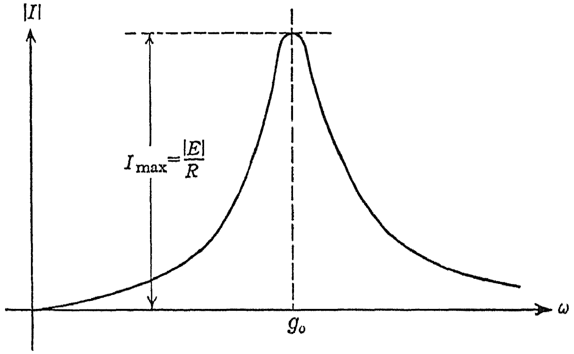


FIG. 47

$$I = \frac{E}{R + j\left(L\omega - \frac{1}{C\omega}\right)}. \tag{256a}$$

If we rationalize this, we get:

$$I = \frac{ER}{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2} - j\frac{\left(L\omega - \frac{1}{C\omega}\right)E}{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}. \tag{258}$$

This expression is rather unwieldy to handle. It may be simplified very materially in the following way. Consider the function:

$$f(\omega) = L\omega - \frac{1}{C\omega}. \tag{259}$$

If we can bring this into a more compact form, then (258) will be easier to handle. $f(\omega)$ is the net reactance of the circuit. It becomes zero at the point $\omega = g_0$, but is regular and continuous there. Hence $f(\omega)$ can be expanded into a Taylor's series about the point $\omega = g_0$ in the usual way, thus:

$$f(\omega) = f(g_0) + \frac{1}{1!}f'(g_0)(\omega - g_0) + \frac{1}{2!}f''(g_0)\cdot(\omega - g_0)^2 + \cdots. \tag{260}$$

If we are interested in the value of $f(\omega)$ in the vicinity of g_0, then we may drop terms above the first in $(\omega - g_0)$ with sufficient accuracy. We will find that this suffices for our particular case. Now we see that:

$$f(g_0) = 0$$

and:

$$f'(g_0) = L + \frac{1}{Cg_0^2} = 2L$$

so that we have:

$$f(\omega) = \left(L\omega - \frac{1}{C\omega}\right) \cong 2L(\omega - g_0) \tag{261}$$

for values of ω near g_0.

It may be said in passing that the Taylor expansion is one of the most useful mathematical tools that the physicist or the engineer has at his service. It is used whenever an approximate result is desired in cases where an exact form is both complicated and unnecessary, as in the present problem, for example. If the reader is not already familiar with this method, he should acquire this familiarity without delay.

Another advantage for the present case is gained if we now introduce the following change of variable:

$$(\omega - g_0) = \alpha x \tag{262}$$

where α is the damping factor for the R, L, C circuit, namely:

$$\alpha = \frac{R}{2L}$$

and x is our new variable. Substituting (261) and (262) back into (258), we get for the complex root-mean-square current amplitude:

$$I = \frac{ER}{R^2 + R^2x^2} - j\frac{ERx}{R^2 + R^2x^2}$$

or:

$$I = \frac{E}{R}\left\{\frac{1}{1 + x^2} - \frac{jx}{1 + x^2}\right\}. \tag{263}$$

This is equivalent to:

$$I = I_{\max} \cdot \frac{1 - jx}{1 + x^2} \tag{264}$$

where:

$$I_{\max} = \frac{E}{R}. \tag{265}$$

If we split the complex current amplitude up into its real and imaginary components:

$$I = I_1 + jI_2 \tag{266}$$

and choose the voltage phase ψ equal to zero, then:

$$I_1 = \frac{I_{\max}}{1 + x^2} \tag{267}$$

and:

$$I_2 = \frac{-I_{\max} \cdot x}{1 + x^2}. \tag{268}$$

Further we see that the absolute magnitude of the root-mean-square current is easily obtained from (264) in the usual way, i.e.:

$$|I| = \frac{I_{\max}}{\sqrt{1 + x^2}}. \tag{269}$$

Now let us see what we have gained by all this manipulation. Eqs. (267), (268) and (269) are the interesting ones to discuss.

They are respectively the real and imaginary portions of the complex current amplitude, and its magnitude. In Fig. 48 we show plots of I_1 and I_2 vs. x. Note that the point $x = 0$ corresponds to the resonant point $\omega = g_0$, which follows from eq. (262). The units $x = 1, 2, \ldots$ etc. are $\omega = g_0 + \alpha, g_0 + 2\,\alpha, \ldots$ etc. i.e., we are measuring in units of α to the right and left of the resonant point. This gives us a good idea of what the functions I_1 and I_2 look like in the vicinity of resonance, which is just what we are after. We do not care so much about their behavior far from resonance since we know anyway that they simply become smaller there. We do want to know, however, how fast they drop off in value to either side of the resonant point. The equations used to plot Fig. 48 are admirably suited for just this purpose. They are, of course, inaccurate for values of ω far from resonance, i.e., for large values of x, on account of the fact that we dropped all terms above the linear term in our Taylor expansion (260), but we shall not use them for such large values.

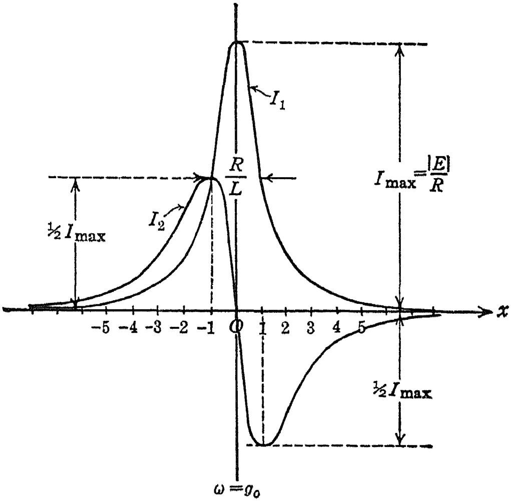


FIG. 48

Getting back to Fig. 48, we see that the real component I_1 is simply a humped curve like the resonance curve. The width of this curve at half its maximum is two units or $2\,\alpha$ radians per second. The imaginary component I_2 has a peculiar shape. It is positive below resonance and negative above. But this is as it should be because the net reactance is capacitive below resonance and inductance above. This means that the current leads the voltage below resonance and lags it above. Hence the positive I_2 below resonance and the negative I_2 above. At resonance the I_2 curve is, of course, zero because the impedance is a pure resistance there.

As the resistance of the circuit is decreased, α decreases, and hence the curves become more critical about the resonance point, because the units $x = 1, 2, \ldots$ etc. become smaller. Also the maximum value of I_1 increases as R decreases. Note further that I_2 equals plus or minus half of I_1 for x equal to minus or plus one unit respectively. These curves give us a good idea of how I_1 and I_2 vary relative to each other near resonance, and hence also give us an idea of how the current phase with respect to the voltage varies in this vicinity. This phase is obviously given by:

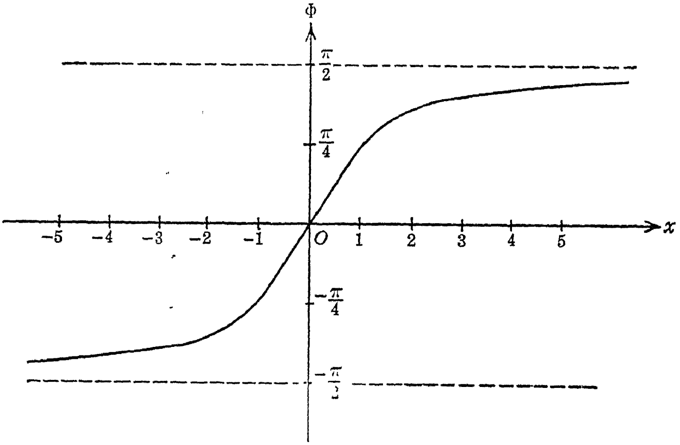


FIG. 49

$$\tan \phi = \frac{I_2}{I_1}$$

or using eqs. (267) and (268) we have:

so that:
$$\left.\begin{aligned} \tan \phi &= -x \\ \phi &= -\tan^{-1} x. \end{aligned}\right\} \qquad (270)$$

A plot of this is given in Fig. 49. Here again we see that a decrease in α (decrease in the size of the x-units), results in a more critical characteristic for the phase angle of the current.

Finally we wish to discuss the eq. (269) for the magnitude of the root-mean-square current. This is plotted in Fig. 50. Comparing this curve with that for I_1 in Fig. 48, we see that it does not fall off as rapidly. The width of $|I|$ at half its maximum value is more than 2 α. The width is 2 α or R/L at 0.707 of the maximum. In most cases this curve for the current magnitude is the one upon which we base the selectivity. The result illustrated in Fig. 50 is interesting in that it brings out the importance of the damping factor α as a measure of the selectivity. There is no particular reason why the width of the resonance curve at 0.707 of its maximum value should be taken as a measure of the selectivity. The width at half the maximum would serve just as well. However, it has become usual practice to specify the selectivity in this way, probably because it takes such a simple form.

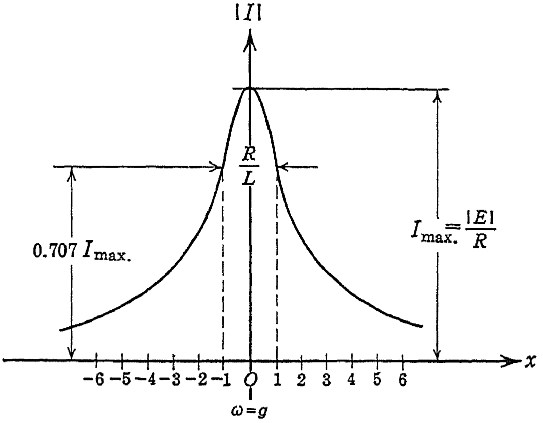


FIG. 50

There is an additional point of interest that has not yet been emphasized in connection with this Fig. 50. Since the selectivity is apparently not dependent upon the capacity in the circuit, it is clear that we can adjust it without affecting the value of g_0, or the resonant frequency which depends only upon the LC product. Thus we see that in order to increase the selectivity of the circuit we should make L as high as possible. This is very definitely limited from practical considerations however. Increasing L will increase the resistance too, unless the wire size is increased. This in turn increases the cost, weight, and size of the coil. The size of L is further limited by the introduction of spurious capacities. Hence this matter of increasing L is not so simple as it appears at first glance.

In this last section we have tried to show how important it is to thoroughly investigate the impedance function with regard to the corresponding behavior of the steady-state current. In this connection another function should be mentioned. The reader has probably already noticed that the steady-state current varies not as the impedance function, but as its reciprocal. In order to emphasize this we often speak of the reciprocal of the impedance function, and call it the **admittance function.** In terms of symbols we have:

$$Y(\omega) = \frac{1}{Z(\omega)}. \tag{271}$$

We shall have more to say about these functions in a later chapter in connection with more complicated networks. We now continue with the general methods whereby these may be attacked, with the view of determining their characteristic behavior.

PROBLEMS TO CHAPTER III

3-1. A voltage function is given by:

$$e = 100 \sin 1000\,t - 50 \cos 1000\,t.$$

Express this as:

(*a*) A sine function with proper phase angle.
(*b*) A cosine function with proper phase angle.
(*c*) The real part of a complex function.
(*d*) The imaginary part of a complex function.
(*e*) The sum of two conjugate complex functions.
(*f*) The difference of two conjugate complex functions.

3-2. The voltage function of Prob. **3-1** is impressed upon an R, L circuit with:

$$L = 10^{-2} \quad R = 10.$$

Determine the steady-state current and express it in all of the forms indicated in Prob. **3-1**.

3–3. Given the function:

$$\begin{aligned} e(t) &= 100 \cos (\omega t + 60^\circ) \\ &= a \sin \omega t + b \cos \omega t \\ &= A \sin (\omega t - \theta) \\ &= (\alpha + j\beta)e^{j\omega t} + (\alpha - j\beta)e^{-j\omega t} \\ &= \mathcal{R}e[Be^{j\omega t}] = \mathcal{R}e[|B|e^{j(\omega t + \phi)}]. \end{aligned}$$

Determine: a, b, A, θ, α, β, B, ϕ. Represent these functions in the complex plane if possible, and also in cartesian coördinates.

3–4. For the function of Prob. **3–3:**

$$i(t) = \frac{d}{dt}\left[\frac{e(t)}{M + jN}\right] = I \sin (\omega t + \delta).$$

Determine I and δ.

3–5. The circuit of Prob. **3–2** is initially at rest. At $t = 0$ a battery of 10 volts is suddenly impressed. At $t = 10^{-3}$ seconds an alternator is suddenly switched in series with the battery. The alternator has an r.m.s. voltage of 6.32 and an angular frequency of 1000 radians per second. At the instant the alternator is introduced, its terminal voltage is zero and increasing negatively. Determine and plot the complete current solution.

3–6. What will be the solution to Prob. **3–5** if the battery is removed at the instant the alternator is inserted?

3–7. The alternator of Prob. **3–5** with the same phase relations at the switching instant, is suddenly impressed upon a resistance of 100 ohms and a capacitance of 10^{-5} farads in series. Determine the initial charge in the condenser necessary to suppress the transient.

3–8. Determine the relative magnitudes of the parameters and driving forces as well as the initial conditions necessary to give rise to a solution in an R, C circuit like that illustrated in Figs. 36 and 37.

3–9. An R, L, C circuit has the following values for its parameters:

$$R = 2 \qquad L = 10^{-3} \qquad C = 10^{-5}.$$

An alternator whose voltage is given by:

$$e(t) = 100 \cos 1000\, t$$

is suddenly impressed at $t = 0$. Initial rest conditions obtain. Determine the complete solution and plot the latter as well as the steady state. Are any excess voltages produced? If so, where do they appear and what are their maximum values? When do these occur?

3–10. Determine the initial conditions necessary to suppress the transient in Prob. **3–9.**

3–11. An R, C circuit contains the following values:

$$R = 100 \qquad C = 10^{-4}.$$

A harmonic voltage given by:

$$e(t) = \mathcal{R}e[Ee^{j\omega t}]$$

with:

$$E = -j\,100 \qquad \omega = 100$$

is suddenly impressed at $t = 0$. Calculate and plot the current as a function of time:

(*a*) When the condenser has an initial charge of $-$ 0.005 coulomb.
(*b*) When the condenser has an initial charge of 0.005 coulomb.

3–12. If the voltage amplitude in Prob. **3–11** is given by:

$$E = 100 \,\underline{|45^\circ}$$

what must be the initial charge in the condenser in magnitude and sign so that the transient will be suppressed?

CHAPTER IV

THE STEADY-STATE SOLUTION FOR THE GENERAL NETWORK

1. Significance of the Problem. In the foregoing chapters we have treated the simplest kind of an electrical network, namely one having but a single mesh. Although this problem is of interest from the standpoint of illustrating the application of fundamental methods of attack, as well as helpful for the interpretation of formal mechanical analogies, it nevertheless lacks the generality necessary to make it useful for practical purposes. Networks found in actual communication work are practically never so simple that the foregoing illustrations suffice for their understanding. Hence it is apparent that in these succeeding chapters the object should be to equip the reader in such a way as to adequately prepare him for anything he might meet in the line of linear electrical networks in practice. This object could be accomplished in more than one way. The more obvious one is to continue with the next case in order from the standpoint of difficulty, then pass on to the next succeeding, and so on until the entire field of possibilities had been exhausted. If we reflect for a moment as to what this would mean in the way of space and time consumption, it will soon become clear even to the beginner that we should have small hopes of ever living long enough to complete our program. In order to illustrate this point, let us mention just a few cases that we would have to consider. In the case of two coupled meshes we would have to take up the following ways in which these may be coupled, viz: By means of self-inductance, mutual inductance, resistance, capacitance, and all the various combinations of these. For each of these, the driving force may be considered: Constant, alternating, a combination of these two, or any number of different frequencies simultaneously impressed. These forces may be considered impressed in the first or the second mesh, or in both at the same time. The desired current may be that in the first or the second mesh. The initial conditions may be arbitrary, and the initial voltage phases may also be arbitrarily chosen. Finally, we should have to consider all possible combinations of the above.

This is just an example of the variety of situations that we face in the case of two coupled meshes. When we pass on to three, and four, etc., the number of special cases increases so rapidly that it is foolish even to think of trying to present the subject in this way. Hence it is clear that we cannot proceed in this fashion. Yet our problem of equipping the student with the necessary tools with which he will be able to attack almost anything that presents itself must be solved. The solution lies in the consideration of what we call the **general network.** This consists of any finite number of meshes linked together in the most general manner conceivable and excited by means of forces of arbitrary form. If we manage to master this case, then obviously all other possible ones will merely be special cases of this one, and will, therefore, require nothing new in the way of fundamental methods of attack.

It may seem that to undertake the solution of this superlatively complicated case at this stage of the presentation is presuming too much on the part of the reader. But he will find that the basic principles involved are no different from those that have already been taken up. The chief difference lies in the larger array of ideas which must be considered simultaneously. This difficulty will be overcome by means of mathematical methods capable of condensing the given array into a field of vision sufficiently small for easy comprehension. Any of these methods with which the reader cannot reasonably be expected to be familiar, will be sufficiently gone into in order to make their use understandable.

2. Network Definitions. First of all, let us make clear to the reader what we mean by a **linear network.** In the preceding chapters we have already seen several instances where a net result was expressed by a sum of individual results. The best example of this is given by the fact that the net behavior of a circuit is equal to the linear addition of the steady-state and transient behaviors. This characteristic is perfectly general in the case of electrical networks under one major condition, namely that the network elements or parameters, i.e., the resistances, inductances, and capacitances, be **constants.** So long as the parameters are constant, the network behavior will be linear in the above sense.

There are, of course, many cases of non-linear networks. A vacuum tube circuit is a non-linear network, because the tube represents, among other things, a resistance which is not constant but varies with the current flowing through the tube. A circuit containing an arc is also non-linear because the arc resistance varies with the current. Another example of a non-linear network is one containing one or more iron-core inductances or transformers. Here the self- or mutual inductances are functions of the current. Other examples are given by microphone circuits where the resistance or impedance forming some branch of the network is varied for the purpose of obtaining an electrical replica of sound impulses. All such cases involving non-constant parameters must be treated by special methods, usually involving approximations to a greater or less extent depending upon the complexity of the situation. They are usually very difficult to solve, and many of them — in fact most of them — are as yet unsolved, at least analytically. They may sometimes be solved graphically or by means of special integrating machines called integraphs. But even then they cannot be attacked without a thorough knowledge of the fundamental methods of attack on linear networks. The latter form the backbone of all subsequent special methods and devices whereby the more difficult network problems are investigated.

Let us next get a clearer picture of what we mean by the term **mesh,** and its relation to the network. One type of network is illustrated in Fig. 51. If we wanted a name to designate this type by, we might call it a **flat network,** meaning thereby that it is possible to flatten it out without having any lines crossing over each other. By meshes we mean the openings in the network like those numbered from 1 to 10 in the figure. They are like the meshes in a fish net, except that the latter are more regular in shape. The boundary of any mesh is called the **mesh-contour,** and consists of the continuous line which a pencil would have to follow in travelling completely around the mesh.

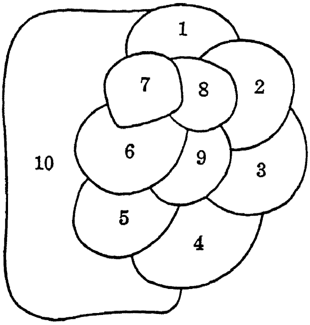


FIG. 51

In this sense, consider the contour of mesh 10 in the Fig. 51. We see that it has portions in common with meshes 1, 7, 6, 5, and 4. Every mesh in the net has one or more of these common portions. We call these, **common branches** or **mutual branches**, and designate them by the two numbers whose meshes they adjoin. Thus the mutual branch 1–8 is that small portion of the contour of mesh 1 which is also part of the contour of mesh 8. Obviously the branch 1–8 is identical with the branch 8–1. This fact seems almost too trivial to mention, but the consequences of this simple fact, as we shall see later, are extremely important in network theory.

The situation illustrated in Fig. 51 is quite easy to understand. But suppose we added one more branch from the midpoint of the branch 5–10 to the midpoint of the branch 2–3 for instance. Then the network would no longer be flat because there would be no way in which to draw this line without crossing over some of the others. It is not hard to see that the addition of this branch would introduce an additional mesh to the network, but its contour is not very obvious. Part of the contour would be the joining-line itself, but the remainder of the contour could be any continuous path through the network joining the two ends of the added branch. The added mesh would then have branches in common with any other meshes whose contours were partly traversed in tracing this path. A similar line of reasoning would apply to any succeeding meshes that were added to the network of Fig. 51 in the above way. Although this may eventually become quite involved, it is nevertheless always physically possible to designate meshes, contours, and common branches in the above manner to any given network.

It should be pointed out that the above procedure is entirely arbitrary. There are innumerable other ways of defining the meshes and designating contours. But the above method seems the simplest and most straightforward, especially in the case of flat networks which are incidentally the most common.

It may sometimes occur that a non-flat network has such a structure that it is hard to see just how many meshes it has. That is, it may be possible to run through the network by means of a certain number of closed paths so as to traverse every branch at least once, and hence we would designate these closed paths as meshes, and say that the network had that many meshes. But how do we know that it may not be possible to find a smaller number of closed paths that will also include each branch at least once? If this were the case, then it would be possible to represent the network by means of fewer individual meshes than our original number. The correct number of meshes whereby to specify a given network is the least number possible, as we shall see later. In the case of a flat network this is as easily done as specifying the meshes in a fish net. In the general case of non-flat networks, the situation is like a chinese puzzle. When there is any doubt about the matter, the following method should be applied.[1]

We may always specify a network completely by means of branches and **branch-points.** The term branch has already been made clear. It is the line joining any two points at which two or more other lines meet. These junction points are called branch-points. The number of branches, as well as the number of branch-points, can always be counted up without question. The least number of meshes in the network is then given by the formula:

$$n = m - \mu + 1 \tag{272}$$

where:

n = least number of meshes in the network.
m = number of branches.
μ = number of branch-points.

Knowing the least number of meshes, these may be specified after a few trials. In order to illustrate the formula (272), let us consider the three-mesh flat network given by Fig. 52. Here $m = 6$ and $\mu = 4$; hence by (272):

$$n = 6 - 4 + 1 = 3$$

which is quite obviously correct. The student need not go very deeply into this matter of determining the least number of meshes in the general case, since in practice it is usually possible to tell by inspection how many meshes there are. We merely give this discussion together with the formula (272) for the sake of completeness. The rigorous proof of (272) has little in it that interests us here, and we therefore omit it entirely.

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[1] W. Feussner, Ann. d. Phys. (4), 9, p. 1304, 1902; (4), 15, p. 385, 1904.

Having defined the meshes, the contours, and the mutual or common branches, we continue with what we call the **mesh-currents.** The latter are numbered consecutively: 1, 2, 3, . . . etc. just like the meshes. Thus i_1, i_2, i_3, . . . etc. represent the mesh-currents for the respective meshes indicated by the subscripts. They are those currents which are assumed to circulate exclusively along the contours of their respective meshes as illustrated by Fig. 52. Their assumed positive direction is indicated by an arrow in each mesh as also shown. This direction may be arbitrarily chosen. In the flat network it is expedient to choose clockwise directions in all the meshes. It is clear from the figure that the total currents in the mutual branches are then always the differences between two mesh-currents. If the network is not flat, the currents in the mutual branches may be the algebraic sum of more than two mesh-currents, since the particular mutual branch in question may be common to more than two meshes. In that case the terms in the algebraic sum may be mixed positive and negative, depending upon the assumed positive directions of the various mesh-currents involved. These are merely matters of convention and may be varied according to individual taste. However, once a certain convention as to positive directions has been chosen, it must be adhered to throughout the work relating to that network. It may be added that in general it is impossible to fix the directions for the mesh-currents in such a way that the mutual branch currents will all be sums of positive terms. There will always be mesh-currents that buck each other in some of the common branches. For the flat network, which is the one most frequently met in practice, the clockwise direction in all meshes is the most straight forward assumption to make. The mutual branch currents are then all numerical differences.

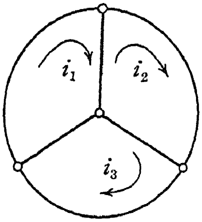

FIG. 52

Sometimes it is more convenient to work with **mesh-charges** instead of mesh-currents. These are merely the time-integrals of the currents, thus:

$$\left.\begin{aligned} q_1 &= \int_0^t i_1\,dt \\ q_2 &= \int_0^t i_2\,dt \\ &\cdots\cdots\cdots\cdots \\ q_n &= \int_0^t i_n\,dt. \end{aligned}\right\} \tag{273}$$

Their use will be illustrated later on.

We come now to the question of network parameters. Before we can settle this we must say more about the kinds of elements which we are going to consider to be present in the various meshes. In general we shall consider that each mesh in the network contains all three kinds of elements, i.e., inductance, resistance, and capacitance, on its contour. Furthermore we are going to assume that each mutual branch contains all three kinds of elements, i.e., all the coupling branches between any two meshes are supposed to be of the most general type. Each mesh is assumed to be coupled to every other mesh in the network by means of resistance, inductance, and capacitance. Fig. 53 illustrates this for a two-mesh network. Here the common inductance consists not only of the self-inductance L_4, but also includes the mutual inductance M. It will be noted that the condensers are designated by S_1, S_2, and S_3. These symbols stand for elastance, which, it will be remembered, is the reciprocal of capacitance. The reason for using elastance in place of capacitance will soon become clear.

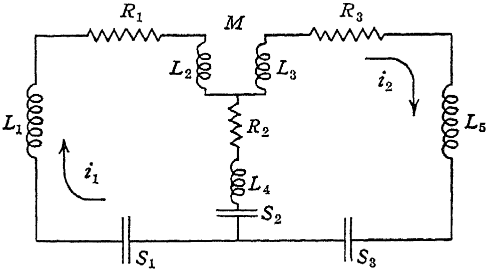

FIG. 53

The best way for us to introduce our method of notation will be to use this simple two-mesh network of Fig. 53 as an illustration. Suppose we wish to write the equilibrium condition for this system. In order to do this we must add up all the counter-voltages that can appear in each mesh, and equate these sums to zero because there are no impressed forces in the meshes. Consider mesh #1 first. On the contour of this mesh we have various inductances, resistances, and elastances. The mesh-current i_1 flowing through these elements produces part of the counter-voltages that we wish to add up. The counter-voltages of self-induction are:

$$(L_1 + L_2 + L_4)\frac{di_1}{dt}\cdot \tag{274}$$

The counter-voltages of resistance are:

$$(R_1 + R_2)i_1. \tag{275}$$

The condenser counter-voltages are:

$$(S_1 + S_2)\int i_1\,dt. \tag{276}$$

We see now why it is better to use elastance rather than capacitance. Elastances add up in series just like resistances or inductances.

The total counter-voltage induced on the contour of mesh #1 by the mesh-current i_1 is the sum of the above three expressions (274), (275), and (276). Here it is expedient to introduce the following notation:

$$\left.\begin{aligned} L_{11} &= L_1 + L_2 + L_4 \\ R_{11} &= R_1 + R_2 \\ S_{11} &= S_1 + S_2. \end{aligned}\right\} \tag{277}$$

The quantities L_{11}, R_{11}, and S_{11} are respectively all the inductance, resistance, and elastance on the contour of mesh #1. They are called the **mesh parameters.** In terms of these the total counter-voltage induced on the contour of mesh #1 by i_1 may be written:

$$L_{11}\frac{di_1}{dt} + R_{11}i_1 + S_{11}\int i_1\,dt. \tag{278}$$

This is not yet all of the counter-voltage induced in mesh #1. The common branch between the meshes carries not only i_1 but i_2 also. The counter-voltage induced there by i_1 is already contained in the expression (278). The mesh-current i_2 also induces a counter-voltage in this common branch which adds linearly to that given by (278) to form the total counter-voltage appearing in mesh #1. We shall now determine this component of counter-voltage induced into mesh #1. That due to the common self-inductance L_4 is:

$$-L_4\frac{di_2}{dt}. \tag{279}$$

The minus sign appears due to the fact that the assumed positive direction for i_2 is opposite to that of i_1. Hence since the counter-voltage:

$$L_4\frac{di_1}{dt}$$

was designated as positive, the counter-voltage:

$$L_4\frac{di_2}{dt}$$

which is fed from mesh #2 into mesh #1 by i_2, must be considered negative. Next we consider the counter-voltage fed from mesh #2 into mesh #1 by i_2 due to the mutual inductance M. This is given by:

$$M\frac{di_2}{dt}. \tag{280}$$

We shall consider the question of the algebraic sign of this term later. The common resistance R_2 causes the introduction of the counter-voltage:

$$-R_2i_2 \tag{281}$$

into mesh #1. Finally, the common elastance S_2 causes:

$$-S_2\int i_2\,dt \tag{282}$$

to appear in mesh #1. The total counter-voltage fed from mesh #2 into mesh #1 by virtue of the common branch 1–2 is given by the sum of the expressions (279), (280), (281), and (282). Before adding these it is convenient to introduce the notation:

$$\left.\begin{aligned} L_{12} &= -L_4 + M \\ R_{12} &= -R_2 \\ S_{12} &= -S_2. \end{aligned}\right\} \tag{283}$$

These quantities L_{12}, R_{12}, and S_{12} are called the **mutual parameters.** Their subscripts indicate to which common branch they belong.

The introduction of the minus signs is a perfectly arbitrary procedure. It is done merely to avoid writing minus signs later. The only rule we need to remember about applying this convention is that if the assumed positive mesh directions, as indicated by the arrows, are opposite in the common branch under consideration, then the common elements are prefixed with negative signs in forming the mutual parameters as in (283). If the arrows coincide in the common branch, then the counter-voltages fed across this branch are positive, and hence the common elements involved remain positive in forming the mutual parameters. With the notation indicated by (283), the total counter-voltage induced into mesh #1 by the current i_2 becomes:

$$L_{12}\frac{di_2}{dt} + R_{12}i_2 + S_{12}\int i_2\,dt. \tag{284}$$

Before continuing, we should say a few words regarding the algebraic sign of the mutual inductance M. From the above it is clear that the purely mutual inductance M plays the same part as a common self-inductance except for the question of algebraic sign. In forming the mutual parameters according to the expression (283) the common inductance, resistance, or elastance is prefixed with a negative or positive sign according to whether the mesh-arrows in this common branch are opposite or coincident respectively, as pointed out above. This same idea cannot be applied to the purely mutual inductance however, because it does not form a physical part of the common branches as do the other parameters. Hence we must be guided entirely by the direction in which voltages are induced. The expression (280) represents a counter-voltage fed from mesh #2 into mesh #1. Its algebraic sign has been assumed positive. This means that a positive mutual inductance must be defined as one which will cause a positive induced counter-voltage in mesh #1 for a positive time rate of change of i_2. This, of course, depends upon the positive assumed directions of both i_1 and i_2, because these fix the positive directions for counter-voltages in the respective meshes. In general, if the mutual inductance between any two meshes causes the introduction of a positive counter-voltage in one mesh for a positive rate of change of current in the other, according to the positive assumed directions, then that mutual inductance is numerically positive with regard to the mutual parameter as expressed by (283). This question regarding the sign of the purely mutual inductance does not arise of course until we have to substitute numerical values in the above equations. In this respect the L's, R's, and S's are always positive numbers and hence never give us any trouble. The M's may, however, be numerically positive or negative according to the above definition. In an actual physical case, measurements must be made to determine the sign of M unless it is obvious from the arrangement of coils and winding senses.

The total counter-voltage appearing in mesh #1 then is the sum of the expressions (278) and (284), so that the condition of dynamic equilibrium for mesh #1 becomes:

$$L_{11}\frac{di_1}{dt} + R_{11}i_1 + S_{11}\int i_1\,dt + L_{12}\frac{di_2}{dt} + R_{12}i_2 + S_{12}\int i_2\,dt = 0. \tag{285}$$

In order to do the same thing for mesh #2, we define:

$$\left.\begin{aligned} L_{22} &= L_3 + L_4 + L_5 \\ R_{22} &= R_2 + R_3 \\ S_{22} &= S_2 + S_3. \end{aligned}\right\} \tag{286}$$

Then the counter-voltage induced on the contour of mesh #2 by the mesh-current i_2 is given by:

$$L_{22}\frac{di_2}{dt} + R_{22}i_2 + S_{22}\int i_2\,dt. \tag{287}$$

The counter-voltage induced into mesh #2 by the current i_1 due to the common branch between meshes #1 and #2 is given by:

$$L_{21}\frac{di_1}{dt} + R_{21}i_1 + S_{21}\int i_1\,dt \tag{288}$$

where obviously:

$$\left.\begin{aligned} L_{21} &= L_{12} \\ R_{21} &= R_{12} \\ S_{21} &= S_{12}. \end{aligned}\right\} \tag{289}$$

The relations (289) should be kept in mind. We will refer to them frequently later on.

The total counter-voltage appearing in mesh #2 is therefore the sum of the expressions (287) and (288), so that the condition of dynamic equilibrium for mesh #2 becomes:

$$L_{21}\frac{di_1}{dt} + R_{21}i_1 + S_{21}\int i_1\,dt + L_{22}\frac{di_2}{dt} + R_{22}i_2 + S_{22}\int i_2\,dt = 0. \tag{290}$$

Equations (285) and (290) together fully describe the force-free equilibrium condition of the two-mesh network illustrated in Fig. 53. The reader may wonder what we have accomplished by all the above symbolism, etc. So far we have accomplished only one thing, and that is to organize the method of procedure for setting up the equilibrium conditions for a network. The object is to put the method in such a form that it is almost fool-proof against getting confused as to algebraic signs and network parameters and currents, no matter how complicated the situation may be. That we have actually accomplished this already will be seen in what follows.

The two-mesh network of Fig. 53 was used merely as a vehicle for the introduction of the defining equations (277), (283), and (286) regarding the mesh and mutual parameters. In the general case we suppose that we have n meshes in our network, where n may be any finite number. We have merely to extend our definitions to this number of meshes in the same sense as they were introduced for the two-mesh case above.

Let us consider any mesh in this n-mesh network, say the kth mesh where k may be any integer between 1 and n. Then the sum of all the inductances on the contour of this mesh will be denoted by L_{kk}, the sum of all the resistances by R_{kk}, and the sum of all the elastances by S_{kk}. These are called the mesh-parameters of the kth mesh.

In this general case that we are considering, each mesh in the network is supposed to be coupled to every other mesh in the net by means of all three kinds of coupling, i.e., inductive, resistive, and elastive coupling, just as in the two-mesh case above. Suppose we consider for instance the mutual branch between mesh k and some other mesh i, different from k. The mutual parameters involved here will be denoted by L_{ik}, R_{ik}, and S_{ik}, or by their equivalents L_{ki}, R_{ki}, and S_{ki} respectively. Here L_{ik} or L_{ki} denotes the sum of all the inductance in the common branch, i.e., common self plus mutual inductance, with the same convention regarding signs as was applied to the two-mesh case above.

Having thus fixed and numbered the meshes and their contours, denoted the positive assumed directions by means of arrows, and determined the mesh and mutual parameters by means of the above notation, we are in a position to proceed with the process of setting up the equations expressing the dynamic equilibrium between impressed and counter-forces in the various meshes of the network.

3. The Equilibrium Conditions for the Network. We shall assume for the present that only one driving force exists in the network, and that this one is located in mesh #1. Later we shall extend our theory to more forces simultaneously impressed. For each mesh we must now add up all the counter-voltages and equate this sum to the impressed voltage in that mesh. Suppose we start with mesh #1. The counter-voltage induced in it due to its own mesh-current is given by:

$$L_{11}\frac{di_1}{dt} + R_{11}i_1 + S_{11}\int i_1\,dt \tag{291}$$

just as in the case of the two-mesh circuit. Mesh #1 is coupled to mesh #2. Hence the counter-voltage induced into mesh #1 from mesh #2 is given by:

$$L_{12}\frac{di_2}{dt} + R_{12}i_2 + S_{12}\int i_2\,dt. \tag{292}$$

Mesh #1 is also coupled to mesh #3. Hence the counter-voltage induced into it by i_3 is:

$$L_{13}\frac{di_3}{dt} + R_{13}i_3 + S_{13}\int i_3\,dt. \tag{293}$$

Similarly, that induced into mesh #1 by i_4 is:

$$L_{14}\frac{di_4}{dt} + R_{14}i_4 + S_{14}\int i_4\,dt. \tag{294}$$

We continue in this way until we come to the counter-voltage which is induced into mesh #1 by the current in the nth (last) mesh. This is:

$$L_{1n}\frac{di_n}{dt} + R_{1n}i_n + S_{1n}\int i_n\,dt. \tag{295}$$

The sum of all the expressions (291) to (295) then represents the total counter-voltage which appears in mesh #1. Equating this to the impressed voltage in mesh #1, we have the equation expressing the condition of dynamic equilibrium of forces in that mesh.

To write this equation out in full would obviously consume much space. We, therefore, need some method of notation whereby this large amount of writing may be compressed into less space. In searching for this, we note that the form of all the expressions (291) to (295) are alike. This suggests the following unified notation. Take, for example, the expression (291). This we may write in the form:

$$\left(L_{11}\frac{d}{dt} + R_{11} + S_{11}\int dt\right)i_1 \tag{291a}$$

by merely factoring out the i_1 which occurs in each term. The three terms in the parenthesis together are called a **differential-integral operator.** They express a certain operation which is to be performed upon i_1, namely that given by (291). This operator could obviously be denoted by a single letter, thus:

$$a_{11} = \left(L_{11}\frac{d}{dt} + R_{11} + S_{11}\int dt\right) \tag{296}$$

where the subscripts on the operator a correspond to those on the mesh parameters. Now the term (291) may be written:

$$a_{11}i_1 \tag{291b}$$

which certainly is much shorter.

Similarly the term (292) may be written in the form:

$$a_{12}i_2 \tag{292a}$$

where:

$$a_{12} = \left(L_{12}\frac{d}{dt} + R_{12} + S_{12}\int dt\right). \tag{297}$$

Thus we continue down to the last expression (295) which becomes:

$$a_{1n}i_n \tag{295a}$$

where:

$$a_{1n} = \left(L_{1n}\frac{d}{dt} + R_{1n} + S_{1n}\int dt\right). \tag{298}$$

The equilibrium condition for mesh #1 in this compressed form therefore becomes:

$$a_{11}i_1 + a_{12}i_2 + a_{13}i_3 + \cdot\cdot\cdot + a_{1n}i_n = e_1 \tag{299}$$

where e_1 is the impressed voltage in mesh #1.

For mesh #2 we must repeat the above procedure identically. The reader will thus see that the equilibrium condition for that mesh becomes:

$$a_{21}i_1 + a_{22}i_2 + a_{23}i_3 + \cdots + a_{2n}i_n = 0 \tag{300}$$

because there are no impressed voltages in any mesh except the first. The operators a have the same significance as for mesh #1 with proper allowance for subscripts. Note that in eq. (300) we have written the term with i_1 first, that for i_2 next, etc. This is for the purpose of preserving order, which is an essential thing in all this work.

It is not hard to see how we should continue. For each mesh in the network we must write an equilibrium equation like the above. In order to bring this whole set of a-operators under a single roof so to speak, we write for them the one defining equation:

$$a_{ik} = \left(L_{ik}\frac{d}{dt} + R_{ik} + S_{ik}\int dt\right) \tag{301}$$

where the indices i and k may take on any integer values from *1* to n. Note that for $i = k$, the operator a_{ii} or a_{kk} involves the mesh parameters L_{ii}, R_{ii}, and S_{ii}, whereas for $i \neq k$, the operator a_{ik} involves the mutual parameters L_{ik}, R_{ik}, and S_{ik}. If we now recall the fact that $L_{ik} = L_{ki}$, $R_{ik} = R_{ki}$, and $S_{ik} = S_{ki}$, we recognize that:

$$a_{ik} = a_{ki} \tag{302}$$

which later will prove a most useful relation.

We are now prepared to write the complete set of equilibrium conditions for the n-mesh network of general form. These are:

$$\left.\begin{array}{l}
a_{11}i_1 + a_{12}i_2 + a_{13}i_3 + \cdots + a_{1n}i_n = e_1 \\
a_{21}i_1 + a_{22}i_2 + a_{23}i_3 + \cdots + a_{2n}i_n = 0 \\
a_{31}i_1 + a_{32}i_2 + a_{33}i_3 + \cdots + a_{3n}i_n = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}i_1 + a_{n2}i_2 + a_{n3}i_3 + \cdots + a_{nn}i_n = 0.
\end{array}\right\} \tag{303}$$

Note the orderly, compact form of these equations. They are as easily written down as the differential equation for a single-mesh circuit. The operators a_{ik} are just as easily determined directly by inspection of the network according to the defining equation (301) and the adopted definitions of the mutual and mesh parameters. The chief feature of this method is its straightforwardness. There is no wasted effort or any chance of going astray throughout the procedure.

It is interesting to note that the physical make-up of the entire network is fully contained in the operators a_{ik}, which may briefly be called the coefficients of the system of equations (303). There is a one-to-one correspondence between the set of coefficients a_{ik} and the network which they represent. By this we mean that if we have given the network, we can write down the set of a_{ik}'s by inspection; and conversely, if we have the set of coefficients given, we can construct the network by inspection.

Note further the peculiarity of the general structure of the system (303). The terms on the diagonal from the upper left to the lower right-hand corner (called the principal diagonal), represent counter-voltages induced in meshes by their own mesh-currents. All the other terms represent counter-voltages which are fed or induced from one mesh into another by virtue of the common branch between these two. These terms may be called "fed-over terms," while those on the principal diagonal may be called "self-induced terms."

An important structural feature of the system (303) is due to the symmetry-condition (302). This causes the system to be symmetrical about the principal diagonal. By this we mean that if we pick out any fed-over term, drop a perpendicular from this point upon the principal diagonal and extend the line an equal distance on the other side, we will locate a term which represents a voltage fed across the same common branch in the opposite direction. The coefficients of these two terms are, of course, equal. This symmetry is perfectly general in the case of linear networks and is useful in many ways.

Finally, we wish to point out that since each individual equation in the system (303) expresses a balance between impressed and counter-electromotive forces, it represents the application of Kirchhoff's e.m.f. law to that mesh. For this reason we say that the system (303) is an expression of Kirchhoff's e.m.f. law applied to instantaneous values, for the network.

Mathematically (303) is a system of simultaneous linear differential equations with constant coefficients. We shall now discuss the application of the heuristic method for obtaining their solution.

4. The Particular Integrals. The chief difference between this general case and that for the simple one-mesh network lies, from the mathematical standpoint, in the fact that now we have a set of simultaneous differential equations, whereas before we had merely one equation with one unknown current. There this single current sufficed to completely specify the condition of the network at all times. Now we must know n currents in order to be able to specify the condition of the network. By the formal analogy to mechanics, we say that our present problem is an n-dimensional one, whereas the former one was one-dimensional. We also say that we are now treating a problem with n degrees of freedom. The actual value of these more or less formal expressions does not appear until we have proceeded much farther than we have at present. Nevertheless it is well for the reader to keep them in mind, and think in terms of these formal analogies as early as possible.

We come now to the process of integrating a system of simultaneous equations such as (303). In order to have something definite to talk about first, let us consider a two-mesh case, where $n = 2$. Then (303) would be:

$$\left.\begin{aligned} a_{11}i_1 + a_{12}i_2 &= e_1 \\ a_{21}i_1 + a_{22}i_2 &= 0. \end{aligned}\right\} \qquad (304)$$

Each equation involves both i_1 and i_2 as unknowns. The elementary way of integrating this pair of equations would be to eliminate one of the currents between the two, and thus obtain a linear differential equation in one variable alone. This could then be solved in a manner similar to that already taken up. Suppose we illustrate this. To eliminate i_2 from (304) we multiply the first equation by a_{22} and the second equation by $-a_{12}$ and add. Note that the operators a_{ik} may in this process be treated very much like ordinary coefficients. This is nothing particularly striking since for example the multiplication of any function by $\frac{d}{dt}$ means the same as differentiating that function with respect to time. This is merely due to the fact that we defined the operators in this way. It might be mentioned here that this process of using operators to indicate differentiation and integration has nothing to do with the so-called operational calculus. Operators were used long before operational calculus was invented. The real essence of the operational calculus lies in the fact that by means of algebraic manipulations of the operators the process of integration is actually carried out, by the help of certain interpretive formulae called operational formulae. The mere use of operators, as we are using them, is not operational calculus, because we are still employing classical mathematics for the process of integration as the reader will see for himself.

Returning to our elimination process above, we have after adding:

$$(a_{11}a_{22} - a_{12}a_{21})i_1 = a_{22}e_1. \tag{305}$$

Note that when multiplying by an operator, the result must always be written so that the operator precedes the function it operates upon. This is just ordinary common sense, of course. Now we note that:

$$\left.\begin{aligned} a_{11}a_{22} &= \left(L_{11}\frac{d}{dt} + R_{11} + S_{11}\int dt\right)\left(L_{22}\frac{d}{dt} + R_{22} + S_{22}\int dt\right) \\ &= L_{11}L_{22}\frac{d^2}{dt^2} + (L_{11}R_{22} + L_{22}R_{11})\frac{d}{dt} \\ &\quad + (L_{11}S_{22} + L_{22}S_{11} + R_{11}R_{22}) + (R_{11}S_{22} + R_{22}S_{11})\int dt \\ &\quad + S_{11}S_{22}\int\int dt\, dt. \end{aligned}\right\} \tag{306}$$

Furthermore:

$$a_{12}a_{21} = \left(L_{12}\frac{d}{dt} + R_{12} + S_{12}\int dt\right)^2 \tag{307}$$

on account of the symmetry condition. Hence:

$$\left.\begin{aligned} a_{12}a_{21} = a_{12}^2 = L_{12}^2\frac{d^2}{dt^2} + 2\,L_{12}R_{12}\frac{d}{dt} + (2\,L_{12}S_{12} + R_{12}^2) \\ + 2\,R_{12}S_{12}\int dt + S_{12}^2\int\int dt\, dt. \end{aligned}\right\} \tag{307a}$$

Thus eq. (305) actually becomes:

$$\left.\begin{aligned} &(L_{11}L_{22} - L_{12}^2)\frac{d^2i_1}{dt^2} + (L_{11}R_{22} + L_{22}R_{11} - 2\,L_{12}R_{12})\frac{di_1}{dt} \\ &\quad + (L_{11}S_{22} + L_{22}S_{11} + R_{11}R_{22} - 2\,L_{12}S_{12} - R_{12}^2)i_1 \\ &\quad + (R_{11}S_{22} + R_{22}S_{11} + 2\,R_{12}S_{12})\int i_1 dt \\ &\quad + (S_{11}S_{22} - S_{12}^2)\int\int i_1\, dt\, dt = L_{22}\frac{de_1}{dt} + R_{22}e_1 + S_{22}\int e_1\, dt. \end{aligned}\right\} \tag{308}$$

In order to eliminate the integral signs from this equation, it is necessary to differentiate it twice with respect to time. The result will then be an inhomogeneous linear differential equation with constant coefficients involving only i_1 and of the fourth order, i.e., involving all time derivatives starting with the fourth. This could then be solved in the ordinary manner as described in Chapter I.

After following the above, the reader will probably agree that this elementary process of solution by elimination is quite laborious. He will note also that the same process of elimination has to be gone through in connection with i_2. Also he should bear in mind that this is illustrated here for the two-mesh case only. What the above method would look like in the case of more meshes we shall leave to his own imagination. Needless to say we shall not continue in this fashion. The reason for giving the above was merely to show how inadequate the elementary methods are in attacking the present situation.

We continue therefore to give the direct method of solution Consider again the two-mesh example for which the pair of equations (304) express the equilibrium condition. We shall assume the driving force in the first mesh to be given by:

$$e_1 = E_1 e^{j\omega t} \tag{309}$$

where the real portion is again understood, and E_1 may be considered as the complex root-mean-square voltage amplitude.

Our first problem is to find the particular integrals or the steady-state solutions for this case. Here we apply the heuristic method again. Since the driving force is harmonic of angular frequency ω, we argue that the steady-state mesh-currents must be of the same form. Hence we assume:

and:

$$\left.\begin{aligned} i_1 &= I_1 e^{j\omega t} \\ i_2 &= I_2 e^{j\omega t} \end{aligned}\right\} \tag{310}$$

where the real portions are, of course, understood, and the I_1 and I_2 are the complex root-mean-square current amplitudes. If the expressions (310) are to be integrals of (304), then they must reduce the latter to identities. We must substitute and see if this is so. Thus we get:

$$\left.\begin{aligned} a_{11}I_1e^{j\omega t} + a_{12}I_2e^{j\omega t} &= E_1e^{j\omega t} \\ a_{21}I_1e^{j\omega t} + a_{22}I_2e^{j\omega t} &= 0. \end{aligned}\right\} \tag{311}$$

Let us see what these are in terms of the network parameters. We have, for example, for the first term of the first equation:

$$\begin{aligned} a_{11}I_1e^{j\omega t} &= I_1\left(L_{11}\frac{d}{dt} + R_{11} + S_{11}\int dt\right)e^{j\omega t} \\ &= I_1\left(L_{11}j\omega + R_{11} + \frac{S_{11}}{j\omega}\right)e^{j\omega t}. \qquad (312) \end{aligned}$$

Similarly for the second term of this equation we have:

$$a_{12}I_2e^{j\omega t} = I_2\left(L_{12}j\omega + R_{12} + \frac{S_{12}}{j\omega}\right)e^{j\omega t}. \qquad (313)$$

Note that the $\frac{d}{dt}$ is now replaced by $j\omega$ while $\int dt$ is replaced by the reciprocal of $j\omega$. The three terms in the parentheses are no longer operators but definitely known functions of ω. However, they still have the same uniform appearance. This suggests the introduction of a new symbol whereby they may be denoted in order to save time and space in writing the equations. Let us put:

$$\left.\begin{aligned} b_{11} &= \left(L_{11}\,j\omega + R_{11} + \frac{S_{11}}{j\omega}\right) \\ \text{and:}\qquad b_{12} &= \left(L_{12}\,j\omega + R_{12} + \frac{S_{12}}{j\omega}\right). \end{aligned}\right\} \qquad (314)$$

Then the first equation (311) becomes:

$$b_{11}I_1e^{j\omega t} + b_{12}I_2e^{j\omega t} = E_1e^{j\omega t}. \qquad (315)$$

In exactly the same way we may obtain for the second equation (311):

$$b_{21}I_1e^{j\omega t} + b_{22}I_2e^{j\omega t} = 0 \qquad (316)$$

where:

$$\left.\begin{aligned} b_{21} &= \left(L_{21}\,j\omega + R_{21} + \frac{S_{21}}{j\omega}\right) \\ \text{and:}\qquad b_{22} &= \left(L_{22}\,j\omega + R_{22} + \frac{S_{22}}{j\omega}\right). \end{aligned}\right\} \qquad (317)$$

Looking at equations (315) and (316), we note that since the factor $e^{j\omega t}$ can never be zero, we can cancel it out, and thus find that the pair of condition equations (311) become:

$$\left.\begin{aligned} b_{11}I_1 + b_{12}I_2 &= E_1 \\ b_{21}I_1 + b_{22}I_2 &= 0. \end{aligned}\right\} \qquad (318)$$

If this pair of condition equations can be satisfied, then the assumed steady-state solutions (310) will be valid and useful. Note that this is a pair of simultaneous algebraic equations with the steady-state current amplitudes I_1 and I_2 as unknowns. It may easily be solved by any of the usual methods, and thus the solutions (310) will be determined.

Note that the coefficients of this pair of equations, given by (314) and (317) are in the form of impedances. For instance, we may write:

$$b_{11} = R_{11} + j\left(L_{11}\omega - \frac{S_{11}}{\omega}\right)$$

which is in the form of:

$$R + jX.$$

Thus we see that b_{11} is the impedance of the contour of mesh #1 for the angular frequency ω. b_{22} is that for the contour of mesh #2, while $b_{12} = b_{21}$ is the impedance of the common branch between meshes 1 and 2. These coefficients are therefore called the **mesh** and **mutual impedances** of the network.

The reader should also realize that these coefficients of (318) could have been written down from inspection of the network, and thus the system (318) could have been set down immediately for the determination of the steady state.

The equations (318) may be given a physical meaning. Each term on the left is the product of an impedance and a root-mean-square current. Hence the terms represent root-mean-square voltages. On the right-hand side we have the root-mean-square impressed voltage. Hence we see that (318) is an expression of the root-mean-square voltage balance for each mesh as contrasted to the corresponding instantaneous voltage balance expressed by the original differential equations (304). We also say that the equations (318) express the time-average equilibrium condition for the network. They are an expression of Kirchhoff's e.m.f. law applied to r.m.s. or steady-state values. If the steady-state solution alone is wanted, then the equations (318) may be written down immediately, and the differential equations (304) omitted entirely. In other words, the time-average instead of the instantaneous equilibrium conditions are set up directly from inspection of the network.

Let us now proceed to a consideration of the general case as expressed by the n-dimensional system (303). In the same way as for the two-mesh case we assume for the steady-state currents:

$$\left.\begin{aligned} i_1 &= I_1 e^{j\omega t} \\ i_2 &= I_2 e^{j\omega t} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ i_n &= I_n e^{j\omega t} \end{aligned}\right\} \tag{319}$$

with the same assumption (309) regarding the impressed voltage. Substitution into (303) gives the system of condition equations:

$$\left.\begin{aligned} b_{11}I_1 + b_{12}I_2 + \cdot\ \cdot\ \cdot + b_{1n}I_n &= E_1 \\ b_{21}I_1 + b_{22}I_2 + \cdot\ \cdot\ \cdot + b_{2n}I_n &= 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ & \cdot\ \cdot \\ b_{n1}I_1 + b_{n2}I_2 + \cdot\ \cdot\ \cdot + b_{nn}I_n &= 0 \end{aligned}\right\} \tag{320}$$

where:

$$b_{ik} = b_{ki} = R_{ik} + j\left(L_{ik}\omega - \frac{S_{ik}}{\omega}\right). \tag{321}$$

The condition (320) which makes the assumptions (319) valid and useful steady-state solutions is nothing more than a system of linear simultaneous algebraic equations in which the complex root-mean-square current amplitudes are the unknowns. By the laws of algebra this condition — or system of conditions — can be satisfied so as to yield the necessary current amplitudes for the assumptions (319).

Before taking up the general method of solving the system (320), let us consider its general structure. Physically it is an expression of Kirchhoff's e.m.f. law applied to steady-state or time-average values. Its external appearance shows that the terms on the principal diagonal are mesh-contour voltages, whereas the balance of the terms represent fed-over voltages which are due to the coupling between each mesh and every other mesh in the network. The condition (321) is again responsible for the symmetry of the system (320) about the principal diagonal.

In carrying out the solution of the system (320), it is almost necessary to make use of the determinantal method. The system can, of course, be solved by ordinary elimination methods, but these become almost prohibitive on account of the amount of space required for demonstrational purposes. Furthermore, elimination processes are very apt to become so entangled as to obliterate the objective entirely, so that nothing of importance may be gained through such a study.

5. The Determinantal Method of Solution. Accordingly we shall give a brief outline of the method of solving linear simultaneous algebraic equations by means of determinants. For a thorough presentation of this subject the reader should refer to a standard text. First, the determinant of the system (320) is defined by the following arrangement of the coefficients:

$$D = \begin{vmatrix} b_{11} & b_{12} \cdot & \cdot & \cdot b_{1n} \\ b_{21} & b_{22} \cdot & \cdot & \cdot b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} \cdot & \cdot & \cdot b_{nn} \end{vmatrix}. \tag{322}$$

The coefficients b_{ik} are called the **elements** of the determinant D. Note that the determinant is merely an arrangement of the coefficients in the same order as to row and column as they appear in the system of equations for which it is written. There are no plus or minus signs between the elements either vertically or horizontally. The method of determining the value of the determinant will be taken up shortly. Note that the first index or subscript of any element denotes the row in which it appears, while the second index denotes the column. Thus b_{57} would be the element at the intersection of the fifth row and seventh column. On account of the symmetry condition:

$$b_{ik} = b_{ki}$$

the network determinant (322) is symmetrical about its principal diagonal. This will always be the case for linear networks.

Next we must explain what we mean by the principal minors of a determinant. A principal minor is also a determinant, but one having one row and column less than the original determinant. It is formed from the latter by cancelling one row and one column, and allowing the remainder to move together so as to fill in the voids left by the cancellation. It is usually denoted by the capital letter corresponding to that used to denote the elements. It is further supplied with the same subscripts or indices as those corresponding to the element located at the intersection of the cancelled row and column. Finally, it is prefixed by an algebraic sign which is positive if the sum of the indices is even and negative if this sum is odd.

Thus we would have for the minor of (322) which is obtained by cancelling the ith row and kth column:

$$B_{ik} = (-1)^{i+k} \cdot \left| \begin{array}{ll:ll} b_{11} & \cdots b_{1(k-1)} & b_{1(k+1)} & \cdots b_{1n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ b_{(i-1)1} & \cdots b_{(i-1)(k-1)} & b_{(i-1)(k+1)} & \cdots b_{(i-1)n} \\ \hdashline b_{(i+1)1} & \cdots b_{(i+1)(k-1)} & b_{(i+1)(k+1)} & \cdots b_{(i+1)n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ b_{n1} & \cdots b_{n(k-1)} & b_{n(k+1)} & \cdots b_{nn} \end{array} \right| \tag{323}$$

The heavy dotted lines indicate the positions of the cancelled row and column. The algebraic sign is controlled by the factor:

$$(-1)^{i+k}$$

which is obviously positive for $i + k$ even and negative for $i + k$ odd. Minors may also be formed by cancelling two or more rows and columns simultaneously. These are called second, third, etc. minors to distinguish them from the principal or first minor described above. They are useful in affording short-cuts in the process of evaluating determinants. The reader should consult the literature on this matter more thoroughly.

We are now prepared to describe one method of evaluating a determinant such as (322). This may be written:

$$D = b_{11}B_{11} + b_{12}B_{12} + \cdots + b_{1n}B_{1n}. \tag{324}$$

The form of this expansion is clear from the above. It follows successively along the terms of the first row. We call this an expansion of D by principal minors along the first row. D may be expanded in this way along any row or column. Thus expanding along the second column, for example, we would have:

$$D = b_{12}B_{12} + b_{22}B_{22} + b_{32}B_{32} + \cdots + b_{n2}B_{n2}. \tag{324a}$$

The numerical result in any case will be the same no matter which row or column we choose to expand along. In a practical example there may be definite preferences in this respect from the standpoint of the simplicity of the result. Thus, if a certain row or column contains a number of zeros, this one would undoubtedly be chosen for the expansion.

Such an expansion reduces the given determinant to a sum of other determinants having one row and column less. Each of these must again be expanded in the above manner. This again yields a sum of determinants which have two rows and columns less than the original. This process is repeated until the result involves nothing but elements. The last step obviously involves the expansion of determinants having only two rows and columns. Such a one is, for example:

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

which yields by expansion:

$$b_{11}\, b_{22} - b_{12}\, b_{21}$$

so that finally there are no determinants left. A single element alone may incidentally be regarded as a determinant having only one row and column.

The above process is a long and tedious one to carry through, as the reader can readily see. In a specific example there is no help for this. There is no shorter method as yet invented which will give the same result.

Just as an illustration let us consider the numerical example:

$$D = \begin{vmatrix} 5 & 3 & 2 \\ 1 & 0 & 7 \\ 6 & 4 & 5 \end{vmatrix}.$$

The second row or column is the one to choose. Take the second row for example, then:

$$\begin{aligned} D &= -1\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} + 0 - 7\begin{vmatrix} 5 & 3 \\ 6 & 4 \end{vmatrix} \\ &= -1(15 - 8) - 7(20 - 18) = -7 - 14 = -21. \end{aligned}$$

We return now to the solution of the system of simultaneous equations (320). We will give no proof, but merely state the so-called Cramer rule for finding the solutions. Thus we have, for example:

$$I_1 = \begin{vmatrix} E_1 & b_{12} \ldots b_{1n} \\ 0 & b_{22} \ldots b_{2n} \\ \cdots & \cdots \\ 0 & b_{n2} \ldots b_{nn} \end{vmatrix} \div D \qquad (325)$$

where D is given by (322). The formation of this result is easily recognizable. It is a quotient whose denominator is the determinant of the system, and whose numerator is a determinant which is obtained from the original one by replacing the elements of the first column by the quantities which appear on the right-hand side of the system (320). The replacement is made for the first column of D because the unknown solved for is the current number one. For current number k we replace the kth column, thus:

$$I_k = \frac{1}{D} \cdot \begin{vmatrix} b_{11} \dots b_{1(k-1)} & E_1 & b_{1(k+1)} \dots b_{1n} \\ b_{21} \dots b_{2(k-1)} & 0 & b_{2(k+1)} \dots b_{2n} \\ \dots\dots & \dots & \dots\dots \\ b_{n1} \dots b_{n(k-1)} & 0 & b_{n(k+1)} \dots b_{nn} \end{vmatrix} . \tag{326}$$

By letting k take on any integer value from 1 to n, we get the corresponding steady-state mesh-currents.

This solution (326) may be put into a more compact form if we recognize the following. Suppose we expand the determinant in the numerator of (326) by principal minors along the kth column. Obviously this expansion will contain only one term because the kth column is composed entirely of zeros except for the first element which is E_1. The minor corresponding to this element, however, is the same as that corresponding to the element b_{1k} in the determinant D. This minor is B_{1k}. Hence (326) may be written:

$$I_k = \frac{E_1 B_{1k}}{D} \tag{326a}$$

which is certainly a compact expression.

By means of this convenient form for the current in any mesh we may easily derive the following interesting relation between the mesh-currents. Suppose we consider the ratio of the current in mesh i to that in mesh k. This is given by:

$$\frac{I_i}{I_k} = \frac{E_1 B_{1i}}{D} \cdot \frac{D}{E_1 B_{1k}} = \frac{B_{1i}}{B_{1k}} . \tag{326b}$$

Thus we see that the ratio of any two mesh-currents is equal to the ratio of the corresponding minors of the network determinant. This result is frequently useful in determining the current in other meshes when that in any one mesh is already known.

6. Generalized Network Impedances. By means of the result (326*a*) we may obtain the steady-state current amplitude in any mesh of the network when the impressed voltage is located in the first mesh. We shall now extend this solution to include cases where the driving force is located in any other mesh of the network.

Let us discuss this in terms of the system of equations (320). If the impressed voltage had been in the second mesh instead of in the first, then this system of equations would have been the same with the one exception, namely that the right-hand members of the equations would all have been zeros except that for the second equation, which would have been E_2. This is clear, if we remember that this system of equations represents the time-average voltage balance in the various meshes. E_2 in this sense represents the root-mean-square impressed voltage amplitude in the second mesh. In general, with the voltage impressed in any mesh, say the ith mesh where i is any integer from 1 to n, the right-hand members would be zero except for that of the ith equation, which would be E_i.

In the solution (326) this means that the kth column contains all zeros except for E_i at the intersection of the ith row. Finally, the form (326*a*) would be the same except for the fact that the numerator would be the product of E_i and the principal minor B_{ik}. Thus for the general case we have:

$$I_k = \frac{E_i B_{ik}}{D}. \qquad (327)$$

This gives the steady-state current in the kth mesh for the impressed voltage E_i in the ith mesh. For $i = 1$ it goes over into the special case (326*a*), of course. The ratio of any two mesh-currents now becomes:

$$\frac{I_s}{I_k} = \frac{B_{is}}{B_{ik}}. \qquad (327a)$$

In the case of the general network it is frequently useful to put the solution in the form of Ohm's law, i.e., to express the current as the impressed voltage divided by an impedance. This impedance for the general case, however, must depend upon where the voltage is impressed and where the current is to be calculated. It must be different for different combinations of these conditions.

This difference may be indicated by means of subscripts. Thus, for example:

$$I_k = \frac{E_i}{Z_{ik}} \tag{327b}$$

would represent the expression for steady-state current in mesh k with the r.m.s. voltage E_i impressed in mesh i. If we compare (327*b*) with (327), we see by inspection that:

$$Z_{ik} = \frac{D}{B_{ik}}. \tag{328}$$

Z_{ik} is called the **generalized network impedance.** It is equal to the network determinant divided by the principal minor corresponding to the indices on the impedance. The generalized network impedances are inversely proportional to the corresponding principal minors. Thus, for example:

$$\frac{Z_{ik}}{Z_{rs}} = \frac{B_{rs}}{B_{ik}}. \tag{328a}$$

By means of this relation any other generalized impedance may easily be calculated when one is known.

7. The Reciprocity Theorem. We remember that the determinant D is symmetrical about its principal diagonal as expressed by the relation (321). Since this is so, it follows by the theory of determinants that the symmetry condition also holds for the principal minors, i.e.:

$$B_{ik} = B_{ki}. \tag{329}$$

The reader may easily convince himself of this by working out a simple case. But a comparison of (328) and (329) shows that the symmetry condition holds also for the generalized network impedances, i.e.:

$$Z_{ik} = Z_{ki}. \tag{330}$$

This is an extremely interesting fact. For if:

$$I_k = \frac{E_i}{Z_{ik}}$$

and:

$$I_i = \frac{E_k}{Z_{ki}}$$

then if:

$$E_i = E_k$$

it follows from (330) that:

$$I_i = I_k.$$

Stated in words this means that if, for example, we impress a generator in mesh #5 of a given network and measure the current, say in mesh #2, and record the result in magnitude and phase position relative to the generator voltage, then if we place the generator in mesh #2 instead of in #5 and measure the current in #5, we will find exactly the same current as before both in magnitude and relative phase. Briefly stated: The source and point of observation may be interchanged without affecting the result. This, of course, assumes that both the source and the observing apparatus have equal impedances, otherwise they would alter the circumstances in the meshes in which they are inserted.

This fact, which is compactly expressed by (330), is called the **reciprocity theorem.** It is an extremely useful theorem in network analysis. Incidentally the above proof of the reciprocity theorem owes its elegance and simplicity entirely to the determinant method of solution, without which it would be exceedingly difficult. By means of the determinant method it follows directly from the almost trivial fact that the common branch between meshes i and k is the same as that between k and i.

8. Input and Transfer Impedance Functions. Consider the generalized network impedance given by the expression (328). Both the determinant D and any of its minors B_{ik} are functions of the elements b_{ik}. The latter in turn are functions of the network parameters L_{ik}, R_{ik} and S_{ik} and the impressed angular frequency ω. Therefore the impedance Z_{ik} is a function of the angular frequency ω. Except for simple cases like those treated in the foregoing chapters, this impedance function is quite complicated. It will be treated in more detail in a later chapter. We wish to point out here the more common forms of these impedance functions.

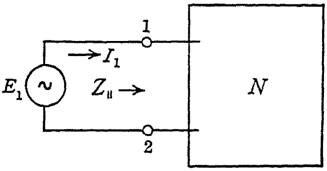

FIG. 54

Suppose we consider the situation illustrated by Fig. 54. Here we have a generator with a terminal voltage equal to E_1 connected to a box N which represents some more or less complicated network. If we know what this network consists of, then we can consider the external loop, which contains the generator, as part of mesh #1. The generator current would then be given by:

$$I_1 = \frac{E_1}{Z_{11}} \tag{331}$$

with:

$$Z_{11} = \frac{D}{B_{11}} \tag{332}$$

where D is the network determinant, considering the external loop as part of mesh #1, and B_{11} is the minor of this determinant obtained by cancelling the first row and column. If we include the generator impedance in that part of mesh #1 beyond the terminals 1–2, then Z_{11} becomes the terminal impedance of the network plus the generator impedance. If the generator impedance is negligible then Z_{11} is the **input impedance** of the network. Another name for Z_{11} is that of **driving-point impedance.** In general the impedance Z_{ik} represents a driving-point impedance when $i = k$, i.e., when the generator is located in the same mesh in which the current is being calculated. The network N is assumed to contain no other sources of electromotive force, and is also assumed to be linear. Under these conditions the value of Z_{11} depends only upon the structure of N and upon its mesh and mutual parameters. N is said to be a **passive network** since the current I_1 depends only upon E_1 and the quantity Z_{11}, which is a function of frequency only.

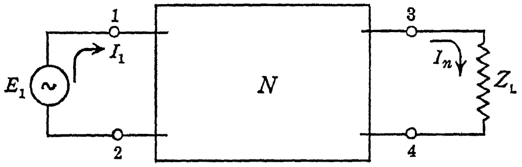

FIG. 55

Suppose on the other hand we consider the situation illustrated in Fig. 55. Here we have a generator with root-mean-square voltage E_1 feeding into a network N at a pair of input terminals 1–2. This network differs from that in Fig. 54 in that it also possesses a pair of output terminals 3–4. It is called a **four-ter-**

minal network to distinguish it from the **two-terminal network** of Fig. 54. In the four-terminal network we are not so much interested in the input current I_1 as we are in the output current I_n, which is that current flowing through the load impedance Z_L. If we consider the generator impedance as part of mesh #1, and Z_L as part of mesh #n of the general network N which is contained in the box, then from the foregoing we can write:

$$I_1 = \frac{E_1}{Z_{11}} \tag{333}$$

while:

$$I_n = \frac{E_1}{Z_{1n}} \tag{334}$$

where:

$$Z_{11} = \frac{D}{B_{11}} \tag{335}$$

and:

$$Z_{1n} = \frac{D}{B_{1n}} \cdot \tag{336}$$

Here D is the determinant of N under the above conditions regarding meshes #1 and #n, while B_{11} and B_{1n} are the principal minors as indicated by their subscripts. Z_{11} is the input or driving-point impedance of the four-terminal network, while Z_{1n} is called the **transfer impedance** because it is a measure of the current which is transferred from terminals 1–2 to terminals 3–4. It is important to note that the transfer impedance as above defined depends not only upon the network N in the box, but also upon the load Z_L as well as upon the generator impedance, i.e., it depends upon the impedances that the four-terminal network works into and out of. This is due to the fact that these impedances form part of the last and first meshes respectively of the complete network. This becomes important in the study of electric wave filters, which are nothing but such four-terminal networks as illustrated in Fig. 55. There the network N, together with its terminal impedances, must be so designed as to give rise to a transfer impedance function Z_{1n} that will vary with frequency in a way which will produce the desired variation of load current I_n. We shall come back to these network matters when we take up the study of wave filters.

It may be mentioned here that it is sometimes more convenient to use what we call the admittance function instead of the impedance function, in the study of two and four-terminal networks. The admittance function is simply the reciprocal of the corresponding impedance function. It represents the current per unit voltage as a function of frequency, just as the impedance function represents the voltage per unit current.

In general the network impedance Z_{ik} represents a transfer impedance whenever $i \neq k$, i.e., when the current response is calculated in mesh #k with the voltage impressed in any other mesh #i, or vice versa.

Finally, it should be noted that if the generator impedance in Fig. 55 equals the load impedance Z_L, then by the reciprocity theorem it will make no difference in the load current at any frequency if the network N is reversed, i.e., if the terminals 1–2 and 3–4 are interchanged.

9. Several Like-Frequency Voltages Simultaneously Impressed. So far we have considered only the response of the network when a single e.m.f. is impressed in one of the meshes. We shall now extend the above theory to such cases where more than one mesh is excited at the same time. For the time being we shall restrict this to the consideration of voltages of like frequency.

Our root-mean-square equilibrium equations then become, instead of (320):

$$\left.\begin{array}{l} b_{11}I_1 + b_{12}I_2 + \cdots + b_{1n}I_n = E_1 \\ b_{21}I_1 + b_{22}I_2 + \cdots + b_{2n}I_n = E_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_{n1}I_1 + b_{n2}I_2 + \cdots + b_{nn}I_n = E_n \end{array}\right\} \tag{337}$$

where $E_1, E_2, \ldots E_n$ are the complex root-mean-square voltage amplitudes simultaneously impressed in meshes $1, 2, \ldots n$ respectively. The solution for any mesh-current then becomes by Cramer's rule, instead of (326):

$$I_k = \frac{1}{D} \cdot \begin{vmatrix} b_{11} \ldots b_{1(k-1)} & E_1 & b_{1(k+1)} \ldots b_{1n} \\ b_{21} \ldots b_{2(k-1)} & E_2 & b_{2(k+1)} \ldots b_{2n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_{n1} \ldots b_{n(k-1)} & E_n & b_{n(k+1)} \ldots b_{nn} \end{vmatrix}. \tag{338}$$

The only difference between this and the solution to the previous case given by (326) is that now the kth column of the determinant in the numerator is completely filled out with voltage amplitudes instead of having just the first term different from zero.

If we now expand this determinant by principal minors along the kth column we shall get n terms instead of just one, namely we have for the numerator of (338):

$$E_1B_{1k} + E_2B_{2k} + \cdot \cdot \cdot + E_nB_{nk}. \tag{339}$$

This may be put into a more compact form by making use of the summation sign, thus:

$$\sum_{i=1}^{n} E_iB_{ik}. \tag{339a}$$

The reader should familiarize himself with this method of notation, since it enables us to write many of the forms that we have to deal with more compactly. This is not done merely for the purpose of saving pencil and paper, but actually enables us to carry out certain operations with ease that otherwise would be obscured by their own bulk. In much of our network theory the mere unwieldiness of certain expressions reminds us of the old saying of "not being able to see the forest for the trees." Whenever this happens it is always expedient to introduce a method of notation which will condense the "trees." The step from (339) to (339a) is one example of such a condensation process. We have already seen others in the nature of symbolic representation. The introduction of the quantities a_{ik} and b_{ik} in the early paragraphs of this chapter were such instances. The engineering student, who is introduced to these methods for the first time, usually dislikes them because they seem to obscure matters rather than clarify them. This initial dislike of symbolic notation will begin to wear off as soon as he sees how much more easily results are obtained by means of it.

Returning to the expression (338), we see that by means of (339a), this becomes:

$$I_k = \frac{\sum_{i=1}^{n} E_iB_{ik}}{D} \tag{340}$$

This is the complex root-mean-square current amplitude in any mesh #k when n like-frequency voltages are impressed simultaneously in the n meshes of the network.

It is interesting to recognize that this result (340) is equivalent to:

$$I_k = \frac{E_1 B_{1k}}{D} + \frac{E_2 B_{2k}}{D} + \cdots + \frac{E_n B_{nk}}{D}. \tag{340a}$$

The first term of this expression represents the current in mesh k resulting from impressing E_1 in mesh #1. The second represents that I_k for E_2 impressed in mesh #2 alone. Finally, the last term represents the response in mesh #k for a voltage E_n impressed in the last mesh. Hence we see that the result of n voltages simultaneously impressed in the various meshes is a linear superposition of the individual responses in mesh #k for voltages $E_1, E_2, \ldots E_n$ impressed successively in meshes $1, 2, \ldots n$ respectively. This is again an excellent example of the superposition principle which holds for linear systems.

10. Voltages of Different Frequencies Simultaneously Impressed. Suppose we demonstrate this case for a simple R, L, C circuit first in order not to confuse the reader with too much symbolism at the start. Our condition of dynamic equilibrium then becomes:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e \tag{341}$$

where:

$$e = E^{(1)}e^{j\omega_1 t} + E^{(2)}e^{j\omega_2 t} \tag{342}$$

i.e., the impressed voltage now consists of the sum of two harmonic voltages of different frequencies, which need not be integer multiples of each other.

Our heuristic method now tells us that for the steady state we should assume that the current will also be composed of a linear sum of harmonic functions of the same frequencies respectively as those given by (342). Thus we assume:

$$i_s = I^{(1)}e^{j\omega_1 t} + I^{(2)}e^{j\omega_2 t}. \tag{343}$$

Substituting the assumption (343) into the differential equation (341) we get:

$$\left(Lj\omega_1 + R + \frac{1}{Cj\omega_1}\right)I^{(1)}e^{j\omega_1 t} + \left(Lj\omega_2 + R + \frac{1}{Cj\omega_2}\right)I^{(2)}e^{j\omega_2 t}$$
$$= E^{(1)}e^{j\omega_1 t} + E^{(2)}e^{j\omega_2 t}. \tag{344}$$

Comparing this with the R, L, C case for a single impressed frequency, we see that the difference lies in the fact that we cannot cancel out the exponential because there are two different frequencies involved. However, it is not hard to see that if the condition (344) is to hold for all values of time, then the coefficients of like powers of e must be equal. Thus (344) splits into the two conditions:

$$\left.\begin{aligned} &\left(Lj\omega_1 + R + \frac{1}{Cj\omega_1}\right)I^{(1)} = E^{(1)} \\ \text{and:}\quad &\left(Lj\omega_2 + R + \frac{1}{Cj\omega_2}\right)I^{(2)} = E^{(2)}. \end{aligned}\right\} \tag{344a}$$

From these the assumed steady-state current amplitudes are easily determined as:

$$\left.\begin{aligned} I^{(1)} &= \frac{E^{(1)}}{R + j\left(L\omega_1 - \dfrac{1}{C\omega_1}\right)} \\ \text{and:}\quad I^{(2)} &= \frac{E^{(2)}}{R + j\left(L\omega_2 - \dfrac{1}{C\omega_2}\right)}. \end{aligned}\right\} \tag{345}$$

This result shows that the superposition principle holds for different frequencies as well as for various voltages of the same frequency. The result for a voltage of the form (342) is the same as the sum of the individual results for voltages equal to the individual terms of (342). This is true for any number of different frequencies simultaneously impressed, as the reader can convince himself by a process similar to the above.

We shall now consider the general network and show that the same result applies with due allowance for the more complicated network impedances. We go back to the system of linear differential equations (303), and let:

$$e_1 = E_1^{(1)}e^{j\omega_1 t} + E_1^{(2)}e^{j\omega_2 t}. \tag{346}$$

We must now assume for the mesh-currents, instead of the forms (319):

$$\left.\begin{aligned} i_1 &= I_1^{(1)}e^{j\omega_1 t} + I_1^{(2)}e^{j\omega_2 t} \\ i_2 &= I_2^{(1)}e^{j\omega_1 t} + I_2^{(2)}e^{j\omega_2 t} \\ &\cdots\cdots\cdots\cdots \\ i_n &= I_n^{(1)}e^{j\omega_1 t} + I_n^{(2)}e^{j\omega_2 t}. \end{aligned}\right\} \tag{347}$$

Each mesh-current is now composed of two frequencies linearly superposed. Substituting the assumptions (346) and (347) into the original system (303), we obtain, after some manipulation, the condition equations:

$$\left.\begin{aligned}
&(b_{11}^{(1)}I_1^{(1)} + \cdots + b_{1n}^{(1)}I_n^{(1)})e^{j\omega_1 t} + (b_{11}^{(2)}I_1^{(2)} + \cdots + b_{1n}^{(2)}I_n^{(2)})e^{j\omega_2 t} = \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots = \\
&(b_{n1}^{(1)}I_1^{(1)} + \cdots + b_{nn}^{(1)}I_n^{(1)})e^{j\omega_1 t} + (b_{n1}^{(2)}I_1^{(2)} + \cdots + b_{nn}^{(2)}I_n^{(2)})e^{j\omega_2 t} = \\
&\qquad = E_1^{(1)}e^{j\omega_1 t} + E_1^{(2)}e^{j\omega_2 t} \\
&\qquad = 0 \\
&\qquad = 0
\end{aligned}\right\} \tag{348}$$

where:

$$b_{ik}^{(1)} = R_{ik} + j\left(L_{ik}\omega_1 - \frac{S_{ik}}{\omega_1}\right)$$

and:

$$\left.\begin{aligned} b_{ik}^{(2)} = R_{ik} + j\left(L_{ik}\omega_2 - \frac{S_{ik}}{\omega_2}\right).\end{aligned}\right\} \tag{349}$$

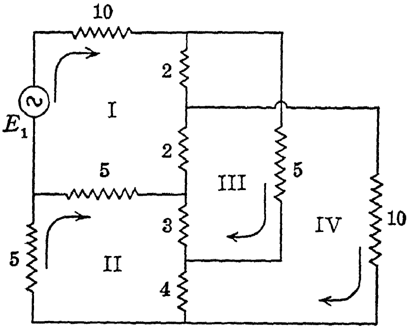

FIG. 56

The conditions (348) can be satisfied for all values of time only when coefficients of like powers of e are equal, i.e., if:

$$\left.\begin{aligned}
b_{11}^{(1)}I_1^{(1)} + \cdots + b_{1n}^{(1)}I_n^{(1)} &= E_1^{(1)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
b_{n1}^{(1)}I_1^{(1)} + \cdots + b_{nn}^{(1)}I_n^{(1)} &= 0
\end{aligned}\right\} \tag{350}$$

and:

$$\left.\begin{aligned}
b_{11}^{(2)}I_1^{(2)} + \cdots + b_{1n}^{(2)}I_n^{(2)} &= E_1^{(2)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
b_{n1}^{(2)}I_1^{(2)} + \cdots + b_{nn}^{(2)}I_n^{(2)} &= 0.
\end{aligned}\right\} \tag{351}$$

The two conditions (350) and (351) are nothing but separate systems expressing the time-average voltage equilibrium conditions for the two frequencies ω_1 and ω_2 respectively. Hence, we see that the superposition principle for different frequencies is justified for the general network.

We shall not extend this to the case where such composite voltages are impressed in all meshes simultaneously, because this is just another superposition as that already shown for n like-frequency voltages simultaneously impressed. In fact the superposition theorem holds generally for linear networks with any distribution of voltages and frequencies simultaneously impressed.

11. Examples. We shall consider now several numerical examples to show how the steady-state system of equations may be set up. In the case of the simpler illustrations we will carry through the solution. In the more involved examples, the problem will merely be set up, and the reader may finish the indicated work for the practice of manipulation.

We start with the simple four-mesh network given by Fig. 56. This is an example of a non-flat network. For simplicity we are assuming that it contains only resistances. The meshes have been numbered in roman numerals so as not to confuse them with the resistance values which are also indicated in the figure. The assumed positive directions for the mesh-currents are given by the arrows. Suppose that the voltage E_1 is given, and the current in mesh IV is desired. From inspection we can write:

$$\begin{aligned}
R_{11} &= 19 & R_{12} &= -5 = R_{21} \\
R_{22} &= 17 & R_{13} &= -4 = R_{31} \\
R_{33} &= 12 & R_{14} &= -2 = R_{41} \\
R_{44} &= 19 & R_{23} &= -3 = R_{32} \\
& & R_{24} &= -7 = R_{42} \\
& & R_{34} &= +5 = R_{43}.
\end{aligned}$$

Hence the determinant of the system becomes:

$$D = \begin{vmatrix} 19 & -5 & -4 & -2 \\ -5 & 17 & -3 & -7 \\ -4 & -3 & 12 & 5 \\ -2 & -7 & 5 & 19 \end{vmatrix} = 43664.$$

Note the symmetry of the determinant. Since we want the transfer impedance Z_{14} we need B_{14}. This is:

$$B_{14} = -\begin{vmatrix} -5 & 17 & -3 \\ -4 & -3 & 12 \\ -2 & -7 & 5 \end{vmatrix} = 479.$$

Thus:

$$Z_{14} = Z_{41} = \frac{43664}{479} = 91.1$$

so that:

$$I_4 = \frac{E_1}{91.1}$$

which is the desired result. Suppose we also want the current I_1. Then we must calculate B_{11}. This is:

$$B_{11} = \begin{vmatrix} 17 & -3 & -7 \\ -3 & 12 & 5 \\ -7 & 5 & 19 \end{vmatrix} = 2902.$$

Hence:

$$Z_{11} = \frac{479}{2902} \cdot 91.1 = 15.05$$

so that:

$$I_1 = \frac{E_1}{15.05}.$$

The current in the first mesh is 6.06 times that in the last. The longest job is that of evaluating the determinant D. The minors are more easily evaluated. Note incidentally that in working out the determinant by expanding along the first row, we have:

$$D = 19\,B_{11} - 5\,B_{12} - 4\,B_{13} - 2\,B_{14}.$$

Hence, all the necessary minors for calculating the current in any mesh are known after D is evaluated. Thus the whole drudgery is over after this is done.

Note the ease with which the problem is set up. It is not necessary to write a single equation. The determinant can be set up almost by inspection of the network, and everything else follows from this. This illustrates the fact that the given network is completely characterized by the determinant of its mesh and mutual impedances.

Next consider the network given in Fig. 57. This consists of nothing but inductances. The purely mutual inductances between the meshes are indicated by the connecting arrows. The values given are in milli-henries. By inspection we can write:

$$\begin{aligned} L_{11} &= 5 \qquad & L_{12} &= -2 + 1 = -1 = L_{21} \\ L_{22} &= 8 & L_{13} &= -1 = L_{31} \\ L_{33} &= 9 & L_{23} &= -3 + 1 = -2 = L_{32}. \end{aligned}$$

Hence:

$$\begin{aligned} b_{11} &= j\omega\, 5 \cdot 10^{-3} \qquad & b_{12} &= -j\omega \cdot 10^{-3} = b_{21} \\ b_{22} &= j\omega\, 8 \cdot 10^{-3} & b_{13} &= -j\omega \cdot 10^{-3} = b_{31} \\ b_{33} &= j\omega\, 9 \cdot 10^{-3} & b_{23} &= -j\omega\, 2 \cdot 10^{-3} = b_{32} \end{aligned}$$

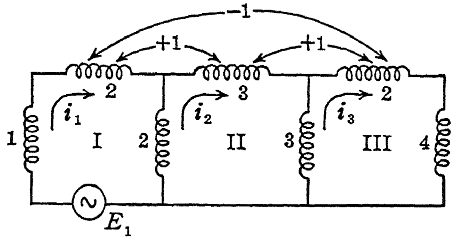

FIG. 57

The determinant of the system is therefore:

$$D = \begin{vmatrix} j\omega\, 5 \cdot 10^{-3} & -j\omega \cdot 10^{-3} & -j\omega \cdot 10^{-3} \\ -j\omega \cdot 10^{-3} & j\omega\, 8 \cdot 10^{-3} & -j\omega\, 2 \cdot 10^{-3} \\ -j\omega \cdot 10^{-3} & -j\omega\, 2 \cdot 10^{-3} & j\omega\, 9 \cdot 10^{-3} \end{vmatrix}$$

or:

$$D = -j\omega^3 \cdot 10^{-9} \cdot \begin{vmatrix} 5 & -1 & -1 \\ -1 & 8 & -2 \\ -1 & -2 & 9 \end{vmatrix} = -j\, 319 \cdot 10^{-9}\omega^3.$$

The principal minors for the first row are:

$$B_{11} = -\omega^2 \cdot 10^{-6} \begin{vmatrix} 8 & -2 \\ -2 & 9 \end{vmatrix} = -68 \cdot 10^{-6}\omega^2$$

$$B_{12} = \omega^2 \cdot 10^{-6} \begin{vmatrix} -1 & -2 \\ -1 & 9 \end{vmatrix} = -11 \cdot 10^{-6}\omega^2$$

$$B_{13} = -\omega^2 \cdot 10^{-6} \begin{vmatrix} -1 & 8 \\ -1 & -2 \end{vmatrix} = -10 \cdot 10^{-6}\omega^2.$$

Hence, the corresponding generalized network impedances are:

$$
\begin{aligned}
Z_{11} &= \frac{-j\,319 \cdot 10^{-9}\omega^3}{-68 \cdot 10^{-6}\omega^2} = j\,0.0047\,\omega \\
Z_{12} &= j\,0.0047\,\omega \cdot \frac{68}{11} = j\,0.029\,\omega \\
Z_{13} &= j\,0.029\,\omega \cdot \frac{11}{10} = j\,0.032\,\omega.
\end{aligned}
$$

If the r.m.s. voltage E_1 is given as:

$$E_1 = 100\,\underline{|0}$$

and:

$$\omega = 2\,\pi f = 1000$$

then:

$$
\begin{aligned}
I_1 &= \frac{100}{4.7\,j} = -j\,21.25 \\
I_2 &= \frac{100}{29\,j} = -j\,3.44 \\
I_3 &= \frac{100}{32\,j} = -j\,3.12
\end{aligned}
$$

so that the steady-state mesh-currents become:

$$
\begin{aligned}
i_1 &= \mathcal{R}_e[21.25\,(-j)e^{j\,1000\,t}] = 21.25 \sin 1000\,t \\
i_2 &= \mathcal{R}_e[3.44\,(-j)e^{j\,1000\,t}] = 3.44 \sin 1000\,t \\
i_3 &= \mathcal{R}_e[3.12\,(-j)e^{j\,1000\,t}] = 3.12 \sin 1000\,t.
\end{aligned}
$$

Note that the impedances are all purely inductive reactances. This is to be expected since the network contains nothing but inductances. The above example illustrates how the determinantal method organizes the work. The introduction of various mutual inductances causes no confusion of sign once the definitions have been fixed.

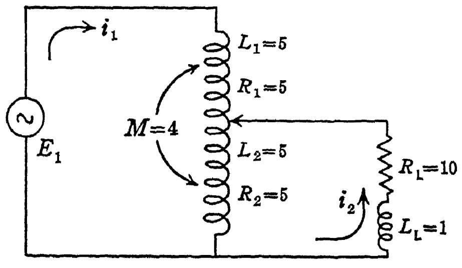

FIG. 58

Another example is illustrated by Fig. 58. This represents an auto-transformer with associated load impedance as indicated. The inductances are given in milli-henries, and the resistances in ohms. This is one case where it is better to choose the assumed positive directions so that the arrows coincide in the common branch. If we do this, then the numerical sign of M is the same in the expression for total primary inductance as it is for the mutual parameter. The reader will appreciate this if he recognizes that with the above choice of positive directions, a positive $\frac{di_1}{dt}$ coincides in direction with a positive $\frac{di_2}{dt}$ in the common branch, so that the induced voltages due to M will have the same sign whether they are due to i_1 or i_2. For this reason M will have the same numerical sign when associated with i_2 as it has when associated with i_1. To determine the sign of M in this case is quite simple. If the lower part of the primary winding is a continuation of the upper part in the same winding sense, then obviously the direction for a positive increasing flux will be the same in both parts, so that M becomes numerically positive. This is the usual arrangement in an auto-transformer.

In general it is not so easy to determine whether M is positive or negative under the assumed current directions, especially when the two coils in question are separated some distance and stand at an arbitrary angle relative to each other. It may be necessary to determine the answer by test. But whether the case may be reasoned out or not, the fundamental criterion is always this: If, under the assumed positive current directions, a positive rate of change of one of the currents induces a voltage which bucks the flow of the other current, then this voltage is a positive counter-voltage, and hence the M involved is numerically positive, otherwise negative. The two currents involved may sometimes be the same current in different parts of the same mesh-contour as in the case of i_1 in connection with the effect of M upon the total primary inductance in our present problem.

To continue then, we have by inspection of Fig. 58:

$$\begin{aligned} L_{11} &= 5 + 5 + 2\cdot 4 = 18 & L_{12} &= 5 + 4 = 9 = L_{21} \\ L_{22} &= 5 + 1 = 6 & R_{12} &= 5 = R_{21} \\ R_{11} &= 5 + 5 = 10 \\ R_{22} &= 5 + 10 = 15 \end{aligned}$$

where the inductances are in milli-henries. The mesh and mutual impedances are therefore:

$$\begin{aligned} b_{11} &= 10 + j\,18\,\omega \cdot 10^{-3} \\ b_{22} &= 15 + j\,6\,\omega \cdot 10^{-3} \\ b_{12} &= 5 + j\,9\,\omega \cdot 10^{-3} \end{aligned}$$

The determinant of the system has two rows and columns, thus:

$$D = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}{}^2.$$

$$B_{11} = b_{22} \qquad B_{12} = -b_{21}.$$

Hence:

$$I_1 = \frac{E_1 b_{22}}{D} \qquad I_2 = -\frac{E_1 b_{12}}{D}.$$

If E_1 be taken as standard phase, i.e., with zero phase angle, then the angles of I_1 and I_2 are phase angles with respect to the voltage. The student may complete the numerical substitution indicated above.

Finally, let us consider the three-mesh network shown in Fig. 59. We are giving this example to illustrate the process of setting up the network equations in a more complicated case. The values of the parameters are given in ohms, henries, and reciprocal farads — also called darafs. The generator is shown in the common branch 1–3. Let its e.m.f. be given in the usual way by:

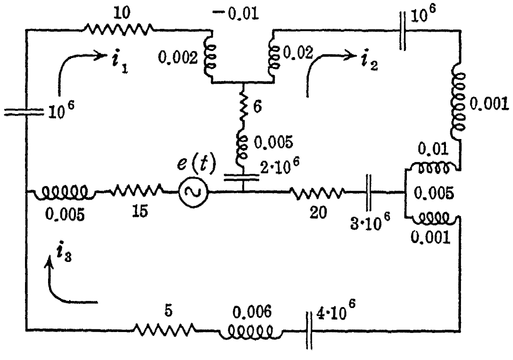

FIG. 59

$$e(t) = Ee^{j\omega t}.$$

When the generator is in the common branch this way, then its phase must be specified with respect to one of the two arrows indicating positive directions. Obviously, if the phase is specified with respect to one of them, then it will be π radians greater or less with respect to the other. Let us assume that the phase of $e(t)$ is specified with respect to the arrow in mesh #1.

By inspection we can write for the mesh and mutual parameters:

$$\begin{aligned}
L_{11} &= 0.012 & L_{12} &= -0.015 = L_{21}\\
R_{11} &= 31 & L_{13} &= -0.005 = L_{31}\\
S_{11} &= 3\cdot 10^6 & L_{23} &= +0.005 = L_{32}\\
L_{22} &= 0.036 & R_{12} &= -6 \quad\;\; = R_{21}\\
R_{22} &= 26 & R_{13} &= -15 \quad = R_{31}\\
S_{22} &= 6\cdot 10^6 & R_{23} &= -20 \quad = R_{32}\\
L_{33} &= 0.012 & S_{12} &= -2\cdot 10^6 = S_{21}\\
R_{33} &= 40 & S_{13} &= \quad 0 \quad\; = S_{31}\\
S_{33} &= 7\cdot 10^6 & S_{23} &= -3\cdot 10^6 = S_{32}.
\end{aligned}$$

Hence, the mesh and mutual impedances become:

$$\begin{aligned}
b_{11} &= 31+j\left(.012\,\omega-\frac{3\cdot 10^6}{\omega}\right) & b_{12} &= -6-j\left(.015\,\omega-\frac{2\cdot 10^6}{\omega}\right)=b_{21}\\
b_{22} &= 26+j\left(.036\,\omega-\frac{6\cdot 10^6}{\omega}\right) & b_{13} &= -15-j\,.005\,\omega=b_{31}\\
b_{33} &= 40+j\left(.012\,\omega-\frac{7\cdot 10^6}{\omega}\right) & b_{23} &= -20+j\left(.005\,\omega+\frac{3\cdot 10^6}{\omega}\right)=b_{32}.
\end{aligned}$$

The root-mean-square equilibrium equations are then:

$$\left.\begin{aligned}
b_{11}I_1 + b_{12}I_2 + b_{13}I_3 &= E\\
b_{21}I_1 + b_{22}I_2 + b_{23}I_3 &= 0\\
b_{31}I_1 + b_{32}I_2 + b_{33}I_3 &= -E.
\end{aligned}\right\}$$

The balance of the work is a matter of algebra involving the manipulation of complex quantities. If the result for a fixed ω is desired, then the coefficients b_{ik} become definite complex numbers. Most of the time in solving the above system is consumed in the manipulation of these complex numbers. However, the procedure is highly organized by the above method so that the chances of making fundamental mistakes are reduced to a minimum.

12. Other Methods of Attack. In conclusion we wish to state that there are other methods whereby the general network may be solved. The most important of these proceeds on the basis of branches and branch-points instead of on the mesh basis given above. In this method the branches are numbered consecutively, instead of the meshes. To each branch is assigned a branch-current. Then the mesh-contours are picked out as in the mesh-current method, and to each closed contour the Kirchhoff e.m.f. law is applied. In addition to this, Kirchhoff's current law is applied to all the branch-points except one. This law states in effect that the algebraic sum of all the branch-currents meeting in a branch-point equals zero. Thus if:

m = number of branches
μ = number of branch-points,
n = number of meshes

we will have altogether:

n e.m.f. equations

and:

$\mu - 1$ current equations.

The total number of equations will then be:

$$n + \mu - 1.$$

But by eq. (272) we have

$$n + \mu - 1 = m \tag{272a}$$

so that the total number of equations thus written equals the number of unknown branch-currents. These may then be uniquely solved for by the determinant method given above.

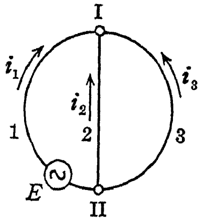

FIG. 60

Let us illustrate this procedure with the simple network given by Fig. 60. Here we have three branches and two branch-points, the latter being numbered by means of roman numerals. The assumed positive directions for the branch-currents are indicated by arrows. Suppose the branch-impedances are given by b_1, b_2, and b_3 respectively. If we pick the branches 1 and 2 to form one mesh-contour, and 2 and 3 to form the other, then the e.m.f. equations become:

$$\left.\begin{aligned} b_1i_1 - b_2i_2 + 0 \quad &= E \\ 0 + b_2i_2 - b_3i_3 &= 0. \end{aligned}\right\} \tag{352}$$

In addition we may write one current equation, say at the branch-point I, thus:

$$\dot{i}_1 + \dot{i}_2 + \dot{i}_3 = 0. \tag{352a}$$

Equations (352) and (352*a*) together just suffice to evaluate the three unknown currents $\dot{i}_1$, $\dot{i}_2$, and $\dot{i}_3$. Note that another current equation written for the branch-point II would be:

$$-\dot{i}_1 - \dot{i}_2 - \dot{i}_3 = 0$$

which is the same as (352*a*), and thus contributes nothing new. In general only $\mu - 1$ independent current equations can be written, as stated above.

This method of attack, based upon branches instead of meshes, is applicable to any case just as well as the mesh-method. Some people even prefer it because it does away with current differences or sums in the common branches. However, the purely mutual effects between branches become more involved regarding numerical signs on account of the greater number of assumed positive directions (arrows). The chief disadvantage of the branch-method, however, lies in the larger number of simultaneous equations that have to be solved. In the above example, for instance, we have three equations and three unknowns to solve for, whereas by the mesh-method we would only have had two. If the reader has had any experience with simultaneous solutions, he will appreciate what this difference will mean in more complicated cases. From the educational standpoint, however, it is well to know something about this branch-method. By means of it we may obtain several interesting side-lights upon network theory. The derivation of formula (272) for the least number of meshes in a network is based upon the philosophy of the branch-method. Other theorems and short-cut methods, developed in advanced network analysis, are also due to this method. Hence, it is well for the student to at least know of its existence even though he may have no immediate use for it.

PROBLEMS TO CHAPTER IV

4-1. Complete the steady-state solution to the auto-transformer problem given in this chapter.

4-2. Complete the steady-state solution to the network given by Fig. 59 of this chapter.

4–3. A network consisting of three meshes, contains nothing but capacitance. The mesh and mutual parameters are given by:

$$\begin{aligned} S_{11} &= 3 \cdot 10^8 & S_{12} &= -10^8 \\ S_{22} &= 1.6 \cdot 10^8 & S_{23} &= -5 \cdot 10^7 \\ S_{33} &= 1.7 \cdot 10^8 & S_{31} &= -10^8. \end{aligned}$$

A constant source of 1000 volts is impressed in the common branch 1–2. Construct the network and determine the steady-state voltages across the various condensers. Use mesh-charges instead of mesh-currents.

4–4. In the network of Fig. 57 add the following parameters:

$$S_{22} = 10^6 \qquad S_{33} = 0.5 \cdot 10^6$$

the mutual parameters remaining intact. Determine the steady-state current in all three meshes, and plot these as functions of frequency over a range which will include all the extremal values of the functions.

4–5. For the network of Prob. **4–3** let the constant source be replaced by:

$$e(t) = 1000 \sin 1000\, t.$$

Set up the r.m.s. equilibrium conditions for the mesh-currents and solve. Repeat for the mesh-charges. Check the latter by integrating the current solutions. In these solutions, carry the limit $\omega \to 0$, and thus obtain the solutions for the constant source. Check with the results obtained in **4–3.**

4–6. To the network of Fig. 57, with the modifications indicated in **4–4,** add the parameter:

$$R_{12} = 100.$$

Replace the alternator by a constant source of 1000 volts, and determine the steady-state currents and charges as well as the condenser voltages.

4–7. To the network of Prob. **4–6** apply the voltage given in **4–5**, and determine the current and charge solutions. Carry the limit $\omega \to 0$ and check with the results of Prob. **4–6.**

4–8. A single-mesh circuit consisting of:

$$R = 50 \qquad L = 0.04 \qquad S = 10^6$$

in series, has impressed upon it a driving-force given by:

$$e(t) = \mathcal{R}e[100\underline{|0^\circ}\, e^{j5000t} + 50\underline{|30^\circ}\, e^{j10000t}].$$

Determine the steady-state current and plot it as a function of time.

4–9. What would a d.c. meter in the circuit of Prob. **4–8** read? What would an a.c. meter read? What is the energy expended in the resistance? Check this against the power input as determined from an integration process.

CHAPTER V

THE TRANSIENT SOLUTION FOR THE GENERAL NETWORK

1. Introductory Remarks. Thus far we have shown how the dynamic equilbrium conditions for the general network may be set up in terms of what we defined as the mesh and mutual parameters and the instantaneous mesh-currents. We have also outlined the method for determining the particular integrals or steady-state mesh-current solutions. Just as in the single mesh case, however, this is not yet the complete solution. To form this we must find the natural behavior of the network and add it to the forced behavior. We shall see that the determination of the natural behavior, which is the solution to the corresponding force-free or homogeneous equilibrium conditions, is much more involved, both theoretically and practically. These difficulties lie chiefly in the determination of the natural modes of the system and in the evaluation of the constants of integration or the transient current amplitudes. In cases where the network contains three or four meshes the amount of labor involved in carrying out the solution becomes quite appreciable. For this reason the investigation of the transient behavior of a complicated system is very frequently omitted, and an estimate of the net behavior based upon the results of the steady-state analysis alone. This, however, requires experience and familiarity with network behavior in general. Even then it is hard to make estimates of this nature. Transient effects often play peculiar tricks that are easily overlooked by any such method as the above. It is, therefore, necessary that we become familiar with the method whereby the transient portion of the complete solution may be found, no matter how complicated the circumstances may be.

It is not particularly hard to outline the method of obtaining the transient, i.e., of indicating symbolically the steps involved in arriving at the complete complementary functions. The difficulty we spoke of above is purely one of drudgery — slide rule manipulation, pencil-pushing and the like. So far as understanding the object of these motions, i.e., of grasping the philosophy involved is concerned, this is almost as easy to do for the general case as it is for the simple one-mesh problem already treated. Having mastered the philosophy of the general case, we will have also a better insight into the important features that govern the natural behavior. This insight alone sometimes suffices to answer the questions that we may have relative to a specific problem without actually carrying the work through to an explicit result. These points will be appreciated more fully later on.

2. The Homogeneous Solutions. All the fundamental considerations relative to the network and the corresponding system of equilibrium equations is the same as before. We now consider the force-free system, however, instead of the forced system given by (303). Mathematically this is the homogeneous system of differential equations corresponding to (303). Obviously, this is given by:

$$\left.\begin{array}{l} a_{11}i_1 + a_{12}i_2 + \cdots + a_{1n}i_n = 0 \\ a_{21}i_1 + a_{22}i_2 + \cdots + a_{2n}i_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}i_1 + a_{n2}i_2 + \cdots + a_{nn}i_n = 0 \end{array}\right\} \qquad (353)$$

where the a's are the same differential-integral operators defined by (301), and conform to the symmetry condition (302).

We must now find n complementary functions, one for each mesh-current. These must be integrals of the system (353), i.e., they must simultaneously satisfy this system of differential equations. It would be possible, of course, to apply an elimination process to this set of equations, as was illustrated in the preceding chapter, so as to obtain a single homogeneous equation for each current alone, and solve each separately. This would involve an immense amount of work, as will easily be appreciated. Therefore, we apply the heuristic method of finding the complementary functions directly, just as we did in the case of the particular integrals. Here we are guided by what we know to be the form of the complementary function for the simple case already treated. Thus, we are justified in assuming that the forms:

$$\left.\begin{aligned} i_1 &= J_1 e^{pt} \\ i_2 &= J_2 e^{pt} \\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ i_n &= J_n e^{pt} \end{aligned}\right\} \tag{354}$$

will satisfy the homogeneous system (353). Note that we must give subscripts to the J's so as to indicate what mesh-currents they belong to. The p's are assumed the same for all the currents, i.e., no distinguishing subscript is attached to them. We do not know as yet whether this assumption is justified, but it is the simplest one to make. Hence, if it is satisfactory, we will obtain the simplest form for our solutions. If it is not satisfactory, we will have to try something else. However, there is no *a priori* reason for supposing that it is not satisfactory, so that we are justified in proceeding on this basis for the time being. The p's, of course, are the assumed natural modes of our general network, just as they were for the simple one-mesh case. We trust that we shall be able to determine their values later on.

Thus we proceed to substitute the assumed solutions (354) into the given system (353). In order to illustrate the mechanism of this substitution, let us consider the first term of the first equation as an example. This is:

$$a_{11}i_1$$

or by inserting the significance of a_{11}, we have:

$$L_{11}\frac{di_1}{dt} + R_{11}i_1 + S_{11}\int i_1\,dt$$

for this term. Substituting the first equation (354) into this, we get:

$$\left(L_{11}p + R_{11} + \frac{S_{11}}{p}\right)J_1e^{pt}. \tag{355}$$

Note the similarity between this expression and (312) which was the result of substituting the assumed steady-state solution for i_1 into this same term. The only difference is that p replaces $j\omega$, and J_1 replaces I_1. The reader should keep this similarity in mind. It will become useful later on.

In exactly the same way the second term of the first equation in the set (353) becomes:

$$\left(L_{12}p + R_{12} + \frac{S_{12}}{p}\right)J_2 e^{pt}. \tag{356}$$

It is not hard to see that every term in the given set of equations has this general form, the difference being only in the subscripts which always correspond to those on the a's. Hence, it becomes effective to introduce the notation:

$$c_{ik} = L_{ik}p + R_{ik} + \frac{S_{ik}}{p} \tag{357}$$

where i and k may take on any integer values from 1 to n. Obviously the symmetry condition also holds for these new coefficients, i.e.:

$$c_{ik} = c_{ki}. \tag{358}$$

These coefficients c_{ik} are no longer differential or integral operators. They are algebraic polynomials in the variable p. If p is known then the c's are known polynomial coefficients. The complete process of substitution of the set (354) into that given by (353) therefore becomes:

$$\left.\begin{array}{l} c_{11}J_1 + c_{12}J_2 + \cdots + c_{1n}J_n = 0 \\ c_{21}J_1 + c_{22}J_2 + \cdots + c_{2n}J_n = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_{n1}J_1 + c_{n2}J_2 + \cdots + c_{nn}J_n = 0 \end{array}\right\} \tag{359}$$

where the exponential factors have been cancelled out because they can never be equal to zero for any finite values of p or t. The system (359) should now be regarded as a set of condition equations. This result states in effect that the assumptions (354) will be valid and useful provided the conditions (359) can be satisfied in agreement with any other special requirements of the problem such as initial conditions and the like. We must now see whether this is possible.

Suppose we assume for the moment that the p's were known. Then the c's would be known coefficients. If we consider the J's as unknowns, then (359) represents a system of simultaneous algebraic equations. It is a homogeneous system of equations. Presumably it can be solved by elementary methods in terms of the coefficients c_{ik}. These are, of course, as yet unknown, but supposing that they were known, then we would conclude off-hand that the J's could be determined. But these are the transient current amplitudes which heretofore took the place of integration constants. Hence, if the J's could be determined from (359), then we would have a transient solution without integration constants This would be a predicament because we would have no way of meeting our arbitrarily specified initial conditions in the form of initial charges and currents. There must be something that we are overlooking in the above suppositions. We must find the answer to this problem in the theory which governs the existence of solutions to homogeneous systems of simultaneous algebraic equations. This is purely a matter of algebra. It is a branch of algebra with which engineers are usually not familiar, however, so that a brief discussion of it here will be in order.

3. The Solubility Conditions. Consider the system (359) purely in the light of an algebraic set of equations for the time being Suppose we attempted to solve this system for J_1, J_2, . . . etc. by the use of Cramer's rule as outlined in the preceding chapter. For J_1, for example, we would have the quotient of two determinants as usual. The denominator would be the determinant of the system of coefficients:

$$D = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix}. \qquad (360)$$

The numerator would be a determinant which is obtained from this one by replacing the elements of the first column by the right-hand members of the system (359). These are all zeros, however. Hence, the numerator of the expression for J_1 would vanish, so that we would get:

$$J_1 = 0.$$

Similarly also we would get:

$$J_2 = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$J_n = 0.$$

Thus the transient currents, or the complementary functions would all be zero. We see that this result satisfies the given homogeneous system (353) since it reduces each equation to the identity $0 \equiv 0$. This is a mathematically correct solution. It states that the natural behavior of the system may be to remain at rest, i.e., in a state of static equilibrium. These are the so-called trivial solutions for the general network. We met them before in the case of the single-mesh case. We have them again as possible solutions in the general case.

We wish to know, however, whether non-trivial solutions are possible in the general case. If we consider the above result of applying Cramer's rule, we see that the J's need not necessarily come out to be zero if it were possible for the determinant (360) to vanish, i.e., if:

$$D = 0. \tag{361}$$

If this is so, then the expressions for the J's would all take the indeterminate form:

$$J_k = \frac{0}{0}\cdot \tag{362}$$

This result is just what we want. The J's may be finite, but remain as yet undetermined. They may, therefore, take the place of integration constants just as in the simple one-mesh case. Let us follow up this possibility.

Consider what the condition (361) means. The determinant is made up of elements which are themselves polynomials in p as was pointed out before. Since the determinant consists in the end of a sum of terms, each of which is a product of n elements, it is clear that D represents a polynomial in p also, whose degree is equal to n times the degree of an individual element. Since the latter is two, it follows that D is a polynomial of the $2n$th degree in p for the general case. The condition (361) may therefore be written:

$$\Delta_{2n}p^{2n} + \Delta_{2n-1}p^{2n-1} + \cdot\cdot\cdot + \Delta_1 p + \Delta_0 = 0 \tag{361a}$$

where the coefficients Δ are various combinations of the network parameters L_{ik}, R_{ik}, and S_{ik}. This condition eq. (361a) is nothing more than an algebraic equation of the $2n$th degree in p. It is the determinantal equation for the general case. It is usually written in the form:

$$D(p) = 0 \tag{361b}$$

to show that D is a function of p. This is necessary in order to distinguish it from $D(\omega)$ for the steady-state solutions as given by eq. (322).

The determinantal equation serves to determine the natural modes of the system just as it did for the simpler cases discussed previously. These natural modes are the roots of the determinantal equation. They may be real or imaginary or complex. The real values will always be negative for passive networks, since for these the transient solutions must decay with time. The imaginary or complex roots will always come in conjugate pairs, and their real portions will likewise be negative. All this is just as it was for the simpler cases. Thus, we see that the determinantal eq. (361*b*) is the condition for the existence of non-trivial solutions. Its fulfillment determines those values of p which must be used in the assumed solutions (354) in order to make them valid and useful.

4. The Complete Transient Solutions. Thus, we have found that the assumptions (354) are justified in the given form, i.e., with arbitrary amplitudes, and all involving the same value of p. However, since the determinantal equation yields in general $2n$ different values of p, we recognize that we may write altogether $2n$ solutions of the form (354) for each mesh-current. The sum of the $2n$ individual solutions for each current will give the complete transient solution for that mesh. Thus, the forms of the transient mesh-current solutions become:

$$\left.\begin{array}{l} i_1 = J_1^{(1)}e^{p_1t} + J_1^{(2)}e^{p_2t} + \cdots + J_1^{(2n)}e^{p_{2n}t} \\ i_2 = J_2^{(1)}e^{p_1t} + J_2^{(2)}e^{p_2t} + \cdots + J_2^{(2n)}e^{p_{2n}t} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ i_n = J_n^{(1)}e^{p_1t} + J_n^{(2)}e^{p_2t} + \cdots + J_n^{(2n)}e^{p_{2n}t} \end{array}\right\} \qquad (363)$$

where the values $p_1, p_2, \ldots p_{2n}$ are the $2n$ natural modes of the system, i.e., the roots of the determinantal equation.

In order to avoid confusion it is important to note the manner in which the subscripts and superscripts are used. The subscripts on the J's denote the particular mesh-current to which the term belongs. The superscripts denote the mode to which the term belongs, and correspond to the subscript on p in the exponent of the exponential.

5. The Normal Functions of Linear Networks. We see from this result that each transient mesh-current is composed of a linear superposition of terms representing the natural modes of oscillation and decay. It is further important to note that each mesh-current is composed of a linear sum of the same set of modes, or natural "motions." In other words, the same natural frequencies and rates of decay appear in general in all the meshes. The only difference between the various transient mesh-currents is that the amplitudes of the natural modes of behavior differ from mesh to mesh. Since these amplitudes are in general complex, this means that they differ both in magnitude and phase.

Since the individual terms in the expressions (363) represent natural modes of behavior, they are frequently called the natural functions of the system, or also the normal functions. These are very expressive terms since they indicate that the exponential functions are the natural or normal functions by means of which the behavior of the network may be expressed. This terminology has considerable significance in that it identifies the exponential function with the natural behavior of any linear network. This peculiarity of the network is due to its physical build — its inherent properties. These points are stressed here because they may be generalized to the extent of determining large classes of other mathematical functions as being the natural or normal functions peculiar to systems of other physical builds. Thus, it is possible to classify certain mathematical functions into groups of normal functions according to the kinds of physical problems to which they logically give expression. The real utility of these methods of classifying and cataloging functions is found in the study of problems involving continuous media, or distributed parameters. The transmission line problem with uniformly distributed parameters, to which we shall come in a later chapter, serves as an illustration to what we are saying here. Nevertheless it is well for the student to be aware of such matters as early as possible, since it will help him in the general process of orientation and classification which is such a useful aid in grasping and fixing ideas.

It should also be recognized in this connection that the exponential function embraces the circular and hyperbolic functions as special forms belonging to the same family of normal functions.

Other normal functions are, for instance, Bessell's functions and Legendre's functions, etc., which the reader may have met in other fields. These functions all characterize certain physical situations, and are often named accordingly. In this respect Bessell's functions form part of a more general class known as cylinder functions because these are the natural functions for problems involving cylindrical symmetry. Legendre's functions belong to that class known as spherical polynomials because these give expression to problems involving spherical symmetry.

It may seem as though we were going rather astray of our subject in talking about these matters, but unless we give the reader some idea of the meaning of the term normal function, it is best not to introduce it in the first place. The latter procedure would, however, not be consistent with the nature of this presentation which is to consider the matter of network theory in as full a light as possible both mathematically and physically. We spoke in the opening chapter of how the heuristic method of approach operates through the medium of familiarity with classes of functions and the physical systems to which they normally give expression. The reader has seen throughout the treatment of networks so far that the exponential function served to give mathematical expression to all our needs. It is proper therefore that, in the treatment of the natural behavior of the general network, we emphasize this characteristic of the exponential function, and also call attention to the existence of similar situations in other fields which the reader may at some later time be interested in following up.

In close contact with the above line of reasoning, we often refer to the expressions (363) as expansions of the mesh-currents in terms of normal functions. This is not a mere jingle of words, but is intended to convey the following thought. Imagine, if possible, that the electrical network contains certain fictitious meshes which we shall call the **normal meshes,** their number being equal to the number of natural frequencies and rates of decay so that one natural frequency together with its rate of decay may be assigned to each normal mesh. By our mechanical analogies we may also designate these fictitious meshes, in which the normal modes of oscillation and decay are separated out, as the **normal coördinates** of the system, since a coördinate is the mechanical analog of an electrical mesh. The actual meshes are the actual coördinates in this sense. Now we may borrow again from our mechanical picture, and say that in the general case these normal coördinates are so oriented geometrically that each possesses projections upon every actual coördinate. Figuratively speaking, the fictitious normal meshes possess "projections" upon all of the actual meshes. Thus, the resulting "motion" in any one actual coördinate becomes a linear superposition of the projections of the various normal "motions" upon it. This clarifies the result expressed by the set of equations (363) regarding the fact that each mesh-current (actual motion) is composed of a sum of terms representing individually the various natural oscillations and rates of decay peculiar to the entire system or network. It makes this fact seem logical because we can get some sort of a picture of it even though that picture is quite fictitious. The mental construction of such a picture unifies the content of the resulting equations (363), and makes them easy to remember because it lends pictorial significance to them.

This process of visualization does more than merely help retain the results given by the set (363). Since the fictitious normal meshes each contain a transient current possessing only one natural frequency and rate of decay, they isolate the normal modes. This suggests the following question: Is it possible to so construct a network that one or more of the actual meshes become normal ones? That is, is it possible to realize actual physical normal meshes? Is it possible to construct a physical network possessing isolated modes? Such networks may be of practical importance. We shall see in a later chapter that this is possible, and that it does lead to networks with peculiar properties. In advanced network analysis many other benefits are derived from the above reasoning regarding normal modes and normal meshes.

6. The Nature of Modes and Amplitudes. It has already been pointed out that when complex modes occur, they always come in conjugate pairs. We wish to go into this matter somewhat more thoroughly now. Let us illustrate our point with a simple one-mesh network containing R, L, and C. For this case we obtained for the natural modes:

$$p = -\frac{R}{2\,L} \pm \sqrt{\frac{R^2}{4\,L^2} - \frac{1}{LC}} \tag{364}$$

where the two modes p_1 and p_2 are given by the plus and minus sign respectively. We pointed out at the time this case was taken up that the quantity under the radical could be positive, negative, or zero depending upon the relative magnitudes of the parameters. Suppose we consider L and C fixed and see how the modes vary as R is varied continuously. Fig. 61 shows by means of the complex plane how these modes do vary with R. For $R = \infty$, p_1 is zero, and p_2 is at minus infinity which is indicated in the figure as a finite point for convenience. As R decreases, these two modes approach each other as indicated by the arrows. Both modes are negative reals and different from each other until the critical condition:

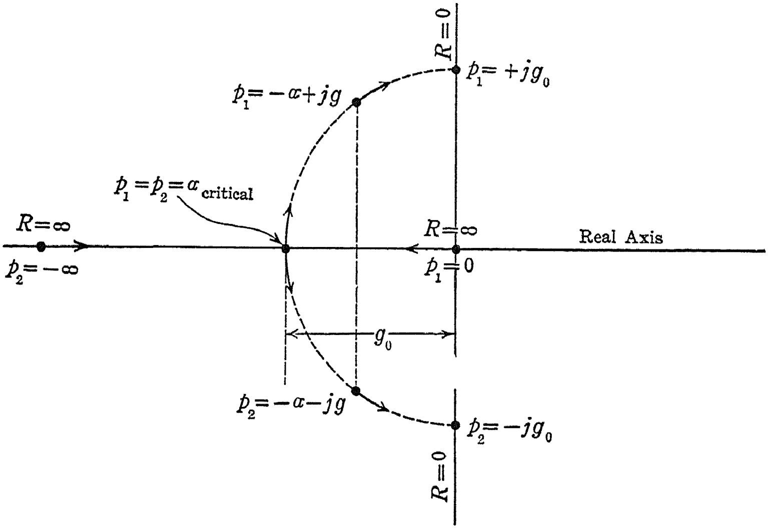

Fig. 61

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

is reached, when they coincide in a single negative real value. When the resistance is further decreased, they again separate, but in such a way as to introduce imaginary portions so that they become complex. It is important to note that these imaginary portions remain equal and opposite, and that the real portions remain equal and negative. It is easily shown that with further decreasing R the modes move along the circumference of a circle with radius equal in magnitude to the coincident modes:

$$\alpha_{\text{critical}} = g_0 = \frac{1}{\sqrt{LC}}.$$

Finally for $R = 0$, the two modes reach the imaginary axis so that their real portions vanish, and their imaginary portions assume their maximum values:

$$g = g_0$$

which is the undamped angular frequency.

When the network is more complicated, the manner in which the modes vary with changing resistances is, of course, quite complicated, and has never been investigated in detail. However, the fundamental behavior is known to be similar to that in the above simple case. Namely, the modes are either negative real and different, or negative real and coincident, or complex in pairs of conjugates, or a mixture of the above kinds. By the latter we mean that it is possible for some of the modes of a complicated system to be different negative reals, others to be coincident negative reals, and still others to be in the form of conjugate complex pairs with negative real portions. The reals or the real parts are always negative so that the roots will always lie wholly on the left of the imaginary axis when plotted in the complex plane. Such an array of modes is shown, for example, in Fig. 62. Here we represent six modes. p_1 and p_2 are real and different, p_3 and p_4 are real and coincident, and p_5 and p_6 are a pair of conjugate complexes. For convenience the conjugate complex modes should always be numbered consecutively.

Obviously, when the degree of the determinantal equation is even, then the number of modes is even, so that it is possible for all of them to be complex, there being half as many pairs as the degree of the equation amounts to. If the resistances involved in the network are small, then this is usually the case. In communication networks this case is commonest. We then say that the network is **fully oscillatory.** However, when the determinantal equation is odd, it is impossible for all the modes to be complex because there can be only an even number of complex modes since they must come in pairs. Hence, when the degree of the equation is odd, and the network is fully oscillatory on account of the low resistances, there will still be one negative real mode. Note that we still call such a network fully oscillatory, because it contains the maximum number of oscillatory modes under the circumstances. If the network contains fewer than the maximum number of oscillatory modes, it is said to be **partially oscillatory.** Such a case is illustrated in Fig. 62. It occurs less frequently in communication work.

Let us consider now the corresponding transient current amplitudes that go with these modes, i.e., the amplitudes $J_k^{(s)}$ in the expressions (363). We remember that the superscript on a J always corresponds to the subscript on the mode it is associated with. Consider now a typical pair of terms:

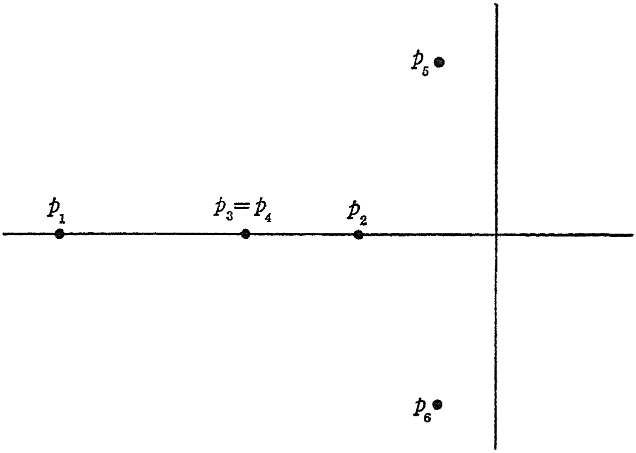

FIG. 62

$$J_k^{(1)}e^{p_1t} + J_k^{(2)}e^{p_2t} \tag{365}$$

which represent a pair of conjugate natural functions in the general expression for the transient current in mesh #k. The modes p_1 and p_2 are assumed to be a pair of conjugate complexes. Since the transient mesh-current, which such pairs of terms represents, must necessarily be a real quantity, it follows that these two terms, taken in their entirety, must be a pair of conjugate complexes, i.e.:

$$J_k^{(1)}e^{p_1t}$$

must be the conjugate of:

$$J_k^{(2)}e^{p_2t}.$$

Since the exponentials are themselves already conjugates, it follows that the corresponding amplitudes $J_k^{(1)}$ and $J_k^{(2)}$ must also be a pair of conjugates. This is necessary in order that the sum (365) be real. For example, if we write:

$$J_k^{(1)} = |J_k^{(1)}|\,\underline{|\phi_k^{(1)}}$$

then we must write for the othe amplitude:

$$J_k^{(2)} = |J_k^{(2)}|\,\underline{|\phi_k^{(2)}} = |J_k^{(1)}|\,\underline{|-\phi_k^{(1)}}$$

so that (365) becomes:

$$|J_k^{(1)}|\{e^{p_1t+j\phi_k^{(1)}} + e^{p_2t-j\phi_k^{(1)}}\}.$$

But the pair of modes are of the form:

$$p_1 = -\alpha + jg$$

and:

$$p_2 = -\alpha - jg$$

so that we have for the sum (365):

$$|J_k^{(1)}|e^{-\alpha t}\left\{e^{j(gt+\phi_k^{(1)})} + e^{-j(gt+\phi_k^{(1)})}\right\}$$

or:

$$2\,|J_k^{(1)}|e^{-\alpha t}\cos\,(gt + \phi_k^{(1)}) \tag{365a}$$

which is a real harmonic function with an angular frequency of g radians per second, a decrement of α per second, an initial amplitude of $2\,|J_k^{(1)}|$ or $2\,|J_k^{(2)}|$ amperes, and a phase angle of $\phi_k^{(1)}$ radians. All this is represented by a pair of terms such as (365), which are written for each pair of conjugate modes in the general expression for the transient current in mesh #k of the network which these modes characterize.

The above discussion should make it clear that the transient current amplitudes, which are the J's in the expressions (363), come in pairs of conjugates when the corresponding modes come in such pairs. In this sense, a pair of conjugate complex amplitudes together define one real amplitude and real phase angle as illustrated by the result (365*a*). This also shows that a pair of conjugate complex modes together define one real angular frequency and one real decrement. On the other hand, the amplitudes of those normal functions for which the modes are real, are themselves real. This follows simply from the fact that the mesh-currents must be real for all values of t.

7. The Number of Modes. It is frequently useful to be able to tell, by inspection of the network, how many modes it possesses, i.e., what the degree of the determinantal equation will be. In the first chapter it was pointed out that this number was not definitely determined by the least number of meshes, i.e., by the degrees of freedom, alone. The least number of meshes, it will be remembered, is defined in general by eq. (272) of Chapter IV in terms of the number of branches and branch-points. In the case of flat networks this number is, of course, evident without the use of a formula. The least number of meshes by which a network may be specified is also called the number of **independent meshes.** This means that the corresponding mesh-currents are independent, i.e., that no one of them can be expressed as a linear combination of the rest. The latter would be possible, of course, if we specified more meshes and mesh-currents than the least number, as the reader can easily illustrate for himself by picking three closed contours in a two-mesh network, for example. Any one of the three currents so specified may be expressed linearly in terms of one or more of the others. This is not possible when only two meshes and currents are specified. More accurately, the least number of meshes in the network are said to be **linearly independent.** We shall have more to say about linear dependence and independence in a later chapter.

Getting back to the subject of the degree of the determinantal equation again, we should be rather pleased if this were related directly to the number of independent meshes or the degrees of freedom of the network. This would make it simple to determine the number of modes that we should expect. However, this is unfortunately not possible. The number of modes depends not only upon the number of independent meshes, but also upon what these meshes contain in the way of coils, condensers, and resistance. The number of meshes do, however, determine the maximum number of modes that the network may contain. This number is twice the number of degrees of freedom, as has been seen while discussing the polynomial form of the determinant $D(p)$. No matter how many coils and condensers are added to the network, the degree of $D(p)$ is not increased above $2n$, where n is the number of independent meshes. This is clear, if we recognize that the highest degree in p of the polynomial elements c_{ik} is two, regardless of the number of coils and condensers in a branch or contour.

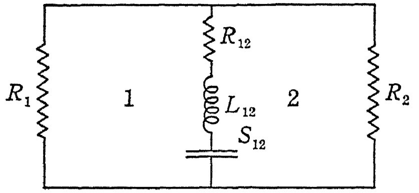

FIG. 63

It is, however, interesting to determine the least number of coils and condensers which will give rise to a determinantal equation of the $2n$th degree. Offhand, it would seem as though this would be the case if the network were so constructed that all the elements c_{ik} were of the second degree in p. This is a necessary but not a sufficient condition. Consider in this connection the network shown in Fig. 63. For this we have:

$$\left.\begin{aligned} c_{12} &= c_{21} = L_{12}p + R_{12} + \frac{S_{12}}{p} \\ c_{11} &= c_{12} + R_1 \\ c_{22} &= c_{12} + R_2 \end{aligned}\right\} \tag{366}$$

so that:

$$D(p) = c_{11}c_{22} - c_{12}^2 = c_{12}(R_1 + R_2) + R_1R_2$$

which is only of the second degree in p although all of the c's given by (366) are of the second degree. However, suppose we modify this network by adding another coil and condenser as shown in Fig. 64. If we again calculate $D(p)$ in the above manner, we will find that it now is of the fourth degree. If we had added only one coil or condenser, then $D(p)$ would have been of the third degree. If we add meshes to the network of Fig. 64 in such a way as to introduce at least one coil and condenser with each additional mesh, then the degree of the determinantal equation will increase by two for each mesh added, and hence will always be equal to $2n$. If one or more of the added meshes fails to introduce its minimum quota of one coil and one condenser, then the degree of $D(p)$ will fall short of $2n$. Another way of putting this is to

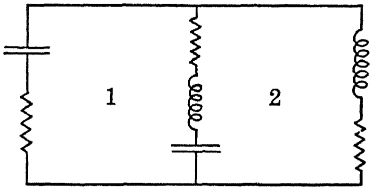

FIG. 64

say that if one independent coil and condenser can be assigned to each mesh in the network, then the number of modes equals $2n$, otherwise less.

If the number of modes is less than $2n$, it is not so simple in all cases to see what the degree of the determinantal equation will be unless the following line of reasoning is applied. We have already seen that the number of integration constants always corresponds to the number of initial conditions that may be specified. This is necessary in order to produce a unique solution. We can therefore consider this fact as axiomatic. Hence, since the number of modes always equals the number of integration constants, or the number of transient current amplitudes for a given mesh-current, their number will also equal the maximum number of initial conditions that we are able to specify. This criterion for determining the degree of the determinantal equation by inspection is easily applied.

For a mesh which contains only capacitance it is clear that nothing but the initial charge can be specified, since the current may be discontinuous at the initial instant. We have already pointed out that such a mesh is called impulsive. For a mesh which contains only inductance, on the other hand, we can specify only the initial current, since there is no place for charge to reside. Finally, for a mesh that contains both inductance and capacitance on its contour, we can specify an initial charge and an initial current. The presence of resistance on the contour of a mesh in addition to capacitance and inductance does not affect the number of initial conditions which we can specify for that mesh. Practically, resistance is always present, of course. We sometimes neglect its presence in order to arrive at an approximate solution of simpler form, but the essential characteristics of the latter are not appreciably changed by the introduction of a reasonable amount of resistance. The resistance can, however, never be neglected in a mesh containing only capacitance, for then the charge could no longer be specified. The current in such a case could become infinite. The reader should bear these things in mind when making approximations. The point which is of importance in the present discussion is that resistance in a mesh does not affect the specification of initial conditions either one way or another. A mesh containing only resistance on its contour does not count at all in the specification of initial conditions.

From this discussion we can tell by inspection how many initial conditions may be specified for a given network. This number then equals the number of modes of the system, or the degree of its determinantal equation. This method should be applied with some caution, however. Consider, for example, the network of Fig. 63. Both meshes have inductance and capacitance on their contours. But we cannot specify four initial conditions because two of these would be the same as the other two, namely the initial charges and the initial currents would have to be the same for the two meshes because the coil and condenser are common to both. The two meshes must contain inductance and capacitance independently before initial charges and currents may be independently assigned to each. Fig. 64 illustrates this case. Here the charge in mesh #1 in no way determines the charge in mesh #2. Also the current in mesh #1 in no way determines that in mesh #2. Consequently the network of Fig. 63 possesses two modes, and that of Fig. 64 possesses four.

This method of determining the number of modes by inspection may be put in the form of a rule, as follows: Assuming that the network is given and the meshes numbered, we begin by following the contour of mesh #1. If it contains an inductance and a capacitance besides resistance, we weight that mesh two. If it contains only inductance or only capacitance besides resistance, we weight it one. If it contains only resistance, we weight it zero. If the contour contains several inductances and capacitances besides resistance, the weight is also two except that in this case only one inductance and capacitance need be checked off in order to give the mesh that weight. In the same way we continue with the other meshes, bearing in mind that a coil or condenser that has already been checked off for a previous mesh does not count. Also, the presence of resistance is not essential unless the mesh contains only capacitance. Then it is manifestly nonsense to neglect resistance. Having thus weighted all the meshes, the total number of modes equals the sum of all the weights. To illustrate this, consider the network of Fig. 65. For mesh #1 we check off the coil and condenser as indicated by the small figures *one* opposite those elements. There are others, but we do not need them in order to give mesh #1 the weight two. The same manner of checking off is used in the remaining meshes. Thus we find:

mesh #	weight
1	2
2	2
3	1
4	0
5	1
6	1
7	2
Total weights	9

The total weights add up to nine. Hence, this network possesses nine modes, and the determinantal equation will be of that degree.

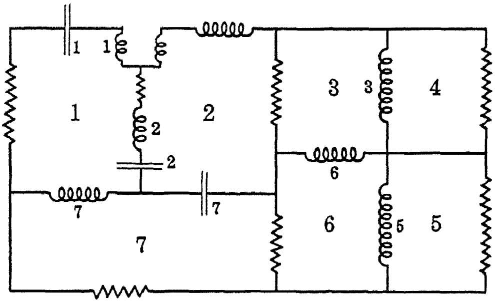

FIG. 65

There will be nine terms in each transient mesh-current solution, and nine integration constants to evaluate.

8. The Relation Between the Amplitudes. Let us return now to the set of transient solutions given by (363). This set contains altogether n times $2n$ terms, and hence also that number of amplitudes represented by the J's. We have rather loosely been speaking of these as integration constants. This is not correct since there can be only $2n$ integration constants at most. From what has just been said regarding the number of modes, we see that very frequently the solutions have considerably less than $2n$ terms each. Whatever the case may be, only the amplitudes for one mesh-current can be considered as the integration constants of the system. It makes no difference which current we select for this rôle. The important point is that, once the J's for one of the mesh-currents is known, the others may be found from these without the use of further initial conditions. This will now be discussed.

For this purpose consider the homogeneous system of equations (359). In this system the J's are the unknowns and the c's are the coefficients. The latter are functions of the modes p. For each value of p, i.e., for each mode, we may determine a whole set of c's. Hence, a homogeneous system of equations like (359) may be written for each mode. For the general network we may therefore write $2n$ such systems as (359), one for each mode. The system for p_1, for example, will involve the amplitudes $J_1^{(1)}$, $J_2^{(1)}$, $\ldots J_n^{(1)}$, where the notation corresponds to that in the set of equations (363). The system for p_2 will involve the amplitudes $J_1^{(2)}, J_2^{(2)}, \ldots J_n^{(2)}$; and so on down to the system for p_{2n} which involves the amplitudes $J_1^{(2n)}, J_2^{(2n)}, \ldots J_n^{(2n)}$.

For the first of these systems of equations the coefficients will be:

$$c_{ik}(p_1) = c_{ik}^{(1)} = L_{ik}p_1 + R_{ik} + \frac{S_{ik}}{p_1}. \tag{367}$$

For the second set they will be given by:

$$c_{ik}(p_2) = c_{ik}^{(2)} = L_{ik}p_2 + R_{ik} + \frac{S_{ik}}{p_2}. \tag{368}$$

For the last set they will be given by:

$$c_{ik}(p_{2n}) = c_{ik}^{(2n)} = L_{ik}p_{2n} + R_{ik} + \frac{S_{ik}}{p_{2n}}. \tag{369}$$

The general coefficient is denoted by:

$$c_{ik}^{(\nu)} = L_{ik}p_\nu + R_{ik} + \frac{S_{ik}}{p_\nu} \tag{370}$$

where i and k take on integer values from 1 to n, and ν from 1 to $2n$. The superscript ν indicates the mode of which the particular coefficient is a function. It always corresponds to the superscript on the J's.

In this notation the homogeneous system (359) for the first mode is written:

$$\left.\begin{array}{l} c_{11}^{(1)}J_1^{(1)} + c_{12}^{(1)}J_2^{(1)} + \cdots + c_{1n}^{(1)}J_n^{(1)} = 0 \\ c_{21}^{(1)}J_1^{(1)} + c_{22}^{(1)}J_2^{(1)} + \cdots + c_{2n}^{(1)}J_n^{(1)} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{n1}^{(1)}J_1^{(1)} + c_{n2}^{(1)}J_2^{(1)} + \cdots + c_{nn}^{(1)}J_n^{(1)} = 0. \end{array}\right\} \tag{371}$$

For the second mode we have:

$$\left.\begin{array}{l} c_{11}{}^{(2)}J_1{}^{(2)} + c_{12}{}^{(2)}J_2{}^{(2)} + \cdots + c_{1n}{}^{(2)}J_n{}^{(2)} = 0 \\ c_{21}{}^{(2)}J_1{}^{(2)} + c_{22}{}^{(2)}J_2{}^{(2)} + \cdots + c_{2n}{}^{(2)}J_n{}^{(2)} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{n1}{}^{(2)}J_1{}^{(2)} + c_{n2}{}^{(2)}J_2{}^{(2)} + \cdots + c_{nn}{}^{(2)}J_n{}^{(2)} = 0 \end{array}\right\} \qquad (372)$$

and so on down to the $2n$th system which is:

$$\left.\begin{array}{l} c_{11}{}^{(2n)}J_1{}^{(2n)} + c_{12}{}^{(2n)}J_2{}^{(2n)} + \cdots + c_{1n}{}^{(2n)}J_n{}^{(2n)} = 0 \\ c_{21}{}^{(2n)}J_1{}^{(2n)} + c_{22}{}^{(2n)}J_2{}^{(2n)} + \cdots + c_{2n}{}^{(2n)}J_n{}^{(2n)} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{n1}{}^{(2n)}J_1{}^{(2n)} + c_{n2}{}^{(2n)}J_2{}^{(2n)} + \cdots + c_{nn}{}^{(2n)}J_n{}^{(2n)} = 0. \end{array}\right\} \qquad (373)$$

This looks like an awful bulk of equations even to think about, but it is really not so bad in this respect. (359), it will be recalled, was a system of condition equations. It was the result of substituting the assumed transient solutions (354) into the original system of differential equations (353). In order to satisfy the conditions (359), and yet make it possible to obtain non-trivial values for the J's, we had to demand that the determinant of the set of coefficients c_{ik} vanish. This demand, expressed by (361), should be looked upon as a second condition equation. It turns out to be an algebraic equation in p of the $2n$th degree. Hence, this second condition can be satisfied not only by one value of p, but by $2n$ values of p which are in general different from each other. These $2n$ p-values we recognize as the modes. Returning then to the conditions (359) we see that, whereas to begin with we had no values of p at all, we now have $2n$ such values each of which will satisfy these conditions, and hence give rise to a set of complementary functions such as (354). The linear sums of $2n$ such functions for each mesh-current therefore form the complete transient solutions as expressed by the set (363). Each of the $2n$ modes gives rise to a particular normal function with its own amplitudes. The amplitudes for a given mode are different in the various meshes. For that mode, say p_1, they are related to each other by a system like (371). For another mode, say p_2, another set of amplitudes for the various meshes obtains, and these are related by the system (372). These sets of amplitudes are those taken by columns in the complete transient solutions (363). Since there are in general $2n$ modes, there must be $2n$ such sets of amplitudes, and hence $2n$ sets of equations which govern their relative complex values in the various meshes, as represented by the sets (371) to (373).

With this in mind consider the set (371) for p_1. If for this set we knew any one of the amplitudes, say that in mesh #1 namely $J_1^{(1)}$, then all the others could be calculated. In order to appreciate this, the reader must know that a system of linear homogeneous algebraic equations, when they are solvable at all, merely give the ratios between the unknowns, but do not give rise to specific solutions. As soon as one of the unknowns is fixed however, then all the others are determined from the above ratios. For our network this means that if the transient amplitude for a particular mode is fixed somehow in one mesh, then the remaining amplitudes will be determined in all the rest of the meshes by the corresponding system of homogeneous equations for that mode, such as (371).

Exactly the same is true for the mode p_2, and so on down to p_{2n}. For each mode the amplitudes in the various meshes are related by ratios which can be determined from the corresponding homogeneous system of equations. For each mode then, we need only determine one amplitude in order to be able to calculate them all. Hence, we need to determine, by some special means, $2n$ amplitudes altogether. Thus, there are in our problem not $2n$ times n arbitrary constants or amplitudes, but only $2n$ of them. This number, however, corresponds to the number of initial conditions which may be specified in the general network of n meshes.

We wish to show now what the ratios between the amplitudes in the various meshes are for a given mode. We saw from (362) that the amplitudes themselves were indeterminate. However, recalling the relation (327a) of Chapter IV for the ratio of any two steady-state current amplitudes, namely:

$$\frac{I_s}{I_k} = \frac{B_{is}}{B_{ik}}$$

we see that this ratio depends only upon minors and not upon the determinant of the system. It will, therefore, not be affected if the latter vanishes. Hence, the same ratio holds for the homogeneous system of equations. Regarding the system (371), for example, we have that:

$$\frac{J_s^{(1)}}{J_k^{(1)}} = \frac{C_{is}^{(1)}}{C_{ik}^{(1)}} \tag{374}$$

where the capital letter $C_{ik}^{(1)}$ represents the minor corresponding to the element $c_{ik}^{(1)}$, and the superscripts relate to the mode p_1. The subscript i relates to a particular row which is arbitrarily chosen. For example, we may write:

$$J_1^{(1)} : J_2^{(1)} : J_3^{(1)} : \cdot \cdot \cdot \cdot : J_n^{(1)} = C_{11}^{(1)} : C_{12}^{(1)} : C_{13}^{(1)} : \cdot \cdot \cdot \cdot : C_{1n}^{(1)}. \tag{375}$$

Or instead of the first row we may pick the second and get:

$$J_1^{(1)} : J_2^{(1)} : J_3^{(1)} : \cdot \cdot \cdot \cdot : J_n^{(1)} = C_{21}^{(1)} : C_{22}^{(1)} : C_{23}^{(1)} : \cdot \cdot \cdot \cdot : C_{2n}^{(1)}. \tag{376}$$

It makes no difference which row we pick, just so long as we keep the same row throughout a series of ratios. This is easily seen from the simple numerical example:

$$\left.\begin{aligned} 3x + 9y &= 0 \\ 2x + 6y &= 0. \end{aligned}\right\}$$

The determinant is:

$$D = \begin{vmatrix} 3 & 9 \\ 2 & 6 \end{vmatrix} = 0.$$

Hence, the system is solvable. Choosing the first row to form minors, we get:

$$x : y = 6 : -2.$$

Choosing the second row, we would have:

$$x : y = -9 : 3$$

which is the same ratio, and also checks with that obtained by transposing one of the terms in either equation. The reader should practice with more complicated numerical examples.

Suppose we choose the first row for the time being. Then we could write instead of the ratios (375):

$$\left.\begin{aligned} J_1^{(1)} &= C_{11}^{(1)}G^{(1)} \\ J_2^{(1)} &= C_{12}^{(1)}G^{(1)} \\ &\cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ J_n^{(1)} &= C_{1n}^{(1)}G^{(1)} \end{aligned}\right\} \tag{377}$$

where $G^{(1)}$ is any constant. This form for the solutions to our homogeneous system (371) is more explicit than (375). It shows that the solutions are not unique, but lack the determination of the constant $G^{(1)}$. The result (377) can be written in the more compact form:

$$J_k^{(1)} = C_{1k}^{(1)}G^{(1)} \qquad (k = 1, 2, \ldots n). \tag{377a}$$

Thus, all the amplitudes corresponding to the mode p_1 are expressed in terms of a single arbitrary constant.

Exactly the same procedure can be followed for the remaining modes. For the rth mode we have:

$$J_k^{(r)} = C_{1k}^{(r)}G^{(r)} \qquad (k = 1, 2, \ldots n). \tag{378}$$

The general expression for any amplitude in the set of equations (363) may be written:

$$J_k^{(r)} = C_{ik}^{(r)}G^{(r)} \qquad \left\{\begin{array}{l} i = \text{arbitrary integer} \\ k = 1, 2, \ldots n \\ r = 1, 2, \ldots 2n \end{array}\right\}. \tag{379}$$

The entire set is determined in terms of the $2n$ arbitrary constants $G^{(1)}, G^{(2)}, \ldots G^{(2n)}$. These are the **true integration constants** of our problem. In an earlier paragraph we mentioned that the $2n$ amplitudes of any one transient mesh-current, i.e., $J_s^{(1)}, J_s^{(2)}, \ldots J_s^{(2n)}$, for example, could be considered as the integration constants. This is merely a matter of personal taste. From (379) we can obviously get:

$$G^{(r)} = \frac{J_s^{(r)}}{C_{is}^{(r)}} \tag{380}$$

where the amplitudes of normal functions for the sth mesh are singled out. Substituting (380) back into (379), we have:

$$J_k^{(r)} = \frac{C_{ik}^{(r)}}{C_{is}^{(r)}} \cdot J_s^{(r)} \qquad \left\{\begin{array}{l} i = \text{arbitrary integer} \\ s = \quad\text{"}\qquad\text{"} \\ k = 1, 2, \ldots n \\ r = 1, 2, \ldots 2n \end{array}\right\}. \tag{381}$$

Thus all the J's are expressed in terms of those for the sth mesh, which may be considered the true integration constants.

The manipulations from (379) to (381) are merely a matter of formality. Any $2n$ quantities could be singled out as integration constants. The important point is that there are but $2n$ arbitrary quantities in the general transient solution for the n-mesh network no matter how we express our solutions symbolically.

To sum up, any network with n degrees of freedom (independent meshes) will have $2n$ or less modes, the same number of integration constants (G's for example), and the same number of initial conditions in the form of mesh charges and currents.

9. Coincident Modes. It is perfectly possible for the determinantal equation to have equal roots, and hence for the network to possess coincident modes. The mathematical side of this question has been treated at the close of the first chapter. We shall show what form this takes in connection with the above general network solution.

Suppose that all the modes from s to $2\,n$ inclusive are coincident, the rest being different from each other. Then the system of transient mesh-current solutions becomes instead of (363):

$$\left.\begin{aligned}
i_1 &= J_1^{(1)}e^{p_1t} + \cdots + J_1^{(s-1)}e^{p_{s-1}t} \\
&\quad + (J_1^{(s)} + J_1^{(s+1)}\cdot t + \cdots + J_1^{(2n)}\cdot t^{2n-s})e^{p_st} \\
i_2 &= J_2^{(1)}e^{p_1t} + \cdots + J_2^{(s-1)}e^{p_{s-1}t} \\
&\quad + (J_2^{(s)} + J_2^{(s+1)}\cdot t + \cdots + J_2^{(2n)}\cdot t^{2n-s})e^{p_st} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
i_n &= J_n^{(1)}e^{p_1t} + \cdots + J_n^{(s-1)}e^{p_{s-1}t} \\
&\quad + (J_n^{(s)} + J_n^{(s+1)}\cdot t + \cdots + J_n^{(2n)}\cdot t^{2n-s})e^{p_st}
\end{aligned}\right\} \qquad (382)$$

where p_s is the single value for all the coincident modes: $p_s = p_{s+1} = \cdots = p_{2n}$. The form of these equations is justified by what was said in the first chapter. We are not interested in giving a rigorous proof for the validity of the set (382) here because there is nothing of importance to be gained by it. The same relations hold between the J's as in the general case. The problem contains the same total number of modes, of integration constants, and initial conditions. The essential difference is in the form of the solutions as indicated by (382).

10. Negative Powers in the Determinantal Equation. The reader will probably have noticed that in forming the determinantal equation it very frequently occurrs that negative powers of p, as well as positive powers, appear. This is due to the fact that the differential equations contain integrals as well as derivatives. Each mesh that contains an independent condenser causes the introduction of an integral due to the fact that the counter-voltage at that condenser is proportional to the corresponding mesh-charge, and hence to the integral of that mesh-current. Since the normal functions are exponentials involving p in the exponent, such integrations produce negative powers of p. In fact the general network which contains independent capacitance in every mesh as well as independent inductance, will give rise to a determinantal equation which involves the same positive and negative powers of p, i.e., the determinantal equation will be a polynomial in p starting with p^n and passing continuously through p^0 to p^{-n}. A simple illustration of this is given by the single R, L, C circuit which gives rise to:

$$D(p) = Lp + R + \frac{1}{Cp}\cdot$$

The determinantal equation for this circuit is:

$$Lp + R + \frac{1}{Cp} = 0. \tag{383}$$

In the previous illustrations in connection with this circuit we simply said that this was equivalent to:

$$Lp^2 + Rp + \frac{1}{C} = 0 \tag{384}$$

and then proceeded to solve this quadratic without justifying the step from (383) to (384). We wish to clarify this point now, and show what the procedure should be in the general case in order to obtain the proper form for the determinantal equation in terms of positive powers of p only.

For this purpose consider again the homogeneous differential equation for the R, L, C circuit. This is:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0. \tag{385}$$

Strictly speaking, this is not a differential equation because it contains an integral also. It can be converted into a differential equation, however, by simply differentiating each term once with respect to time, thus obtaining:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0. \tag{386}$$

It is now ready for integration by the usual method. Assuming:

$$i = Je^{pt}$$

we get:

$$\left(Lp^2 + Rp + \frac{1}{C}\right)Je^{pt} = 0 \tag{387}$$

from which the determinantal equation in the form (384) is evident.

Another way of looking at this situation is to say that since (385) involves an integral, the solution for i must contain a constant of integration for this integral in addition to the constant which enters due to the integration of the differential term. Hence, the solution must contain two constants of integration, and therefore the system must possess two modes, which in turn requires the determinantal equation to be of the second degree

The method just applied to the force-free equilibrium condition (385) for the simple circuit, can also be carried out for the set of homogeneous equations representing the force-free equilibrium of the general network. The trouble is that it cannot be applied to the system (353) directly, except in the general case, for the following reason.

Off-hand it would seem that one merely had to differentiate each equation in the set (353) until it became free from integrals, and then proceed in the usual fashion. This is not always correct. Suppose we consider as an illustration any two-mesh network. For this the homogeneous system (353) becomes:

$$\left.\begin{aligned} a_{11}\dot{i}_1 + a_{12}\dot{i}_2 &= 0 \\ a_{21}\dot{i}_1 + a_{22}\dot{i}_2 &= 0. \end{aligned}\right\} \tag{388}$$

Before we can begin freeing the equilibrium condition from integrals, we must first obtain a single equation involving a single unknown, i.e., we must eliminate between the pair of equations (388). Let us do this so as to retain $\dot{i}_1$. Then, we multiply the first equation by a_{22}, the second by $-a_{12}$, and add, obtaining:

$$(a_{11}a_{22} - a_{12}{}^2)\dot{i}_1 = 0. \tag{389}$$

In the same way we could obtain:

$$(a_{11}a_{22} - a_{12}{}^2)\dot{i}_2 = 0. \tag{390}$$

These are the separate homogeneous equations in i_1 and i_2 alone. Note incidentally that these are simply the determinant of the system (388) multiplied by the respective mesh-current for which the equation is desired. That is if we let:

$$D_d = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{391}$$

where the subscript d denotes that this is a determinant whose elements are differential-integral operators, then (389) and (390) are simply:

$$D_d \cdot i_1 = 0 \tag{389a}$$

and

$$D_d \cdot i_2 = 0 \tag{390a}$$

respectively. This is generally correct for the n-mesh system (353) as the reader can easily demonstrate for himself by means of the above method or by the theory of determinant solutions.

Let us proceed to our point, however, with the two-mesh case as an example. Consider first the network as given by Fig. 66. Here all three kinds of parameters are present in each branch. Therefore:

$$\left.\begin{aligned} a_{11} &= L_{11}\frac{d}{dt} + R_{11} + S_{11}\int dt \\ a_{12} = a_{21} &= L_{12}\frac{d}{dt} + R_{12} + S_{12}\int dt \\ a_{22} &= L_{22}\frac{d}{dt} + R_{22} + S_{22}\int dt \end{aligned}\right\} \tag{392}$$

so that (389), for example, becomes:

$$\begin{aligned} &(L_{11}L_{22} - L_{12}^2)\frac{d^2 i_1}{dt^2} + (L_{11}R_{22} + L_{22}R_{11} - 2\,L_{12}R_{12})\frac{di_1}{dt} \\ &\qquad + (R_{11}R_{22} - R_{12}^2 + S_{11}L_{22} + S_{22}L_{11} - 2\,S_{12}L_{12})\,i \\ &+ (R_{11}S_{22} + R_{22}S_{11} - 2\,R_{12}S_{12})\int i_1\,dt + (S_{11}S_{22} - S_{12}^2)\int\int i_1\,dt\,dt = 0. \end{aligned} \tag{393}$$

The same equation is obtained for i_2. It is clear that two differentiations are necessary in order to free this equation from integrals, thus resulting in a differential equation of the fourth order, so that the determinantal equation becomes one of the fourth degree.

But now let us contrast this result with what we obtain for the two-mesh network illustrated in Fig. 67. Here only one of the meshes contains capacitance independently, i.e., S_{22} and S_{12} are both zero. Introducing this fact into (393), we see that all the terms are retained except the double integral at the end. This would also vanish if $S_{11} = S_{12} = 0$, or if $S_{11} = S_{12} = S_{22}$, i.e., if the only condenser were in the last or the middle branch respectively instead of being in the first. Hence in such a case only one differentiation is necessary in order to clear the resulting equation of integrals although two differentiations would be necessary in the given system (388) if the only condenser were in the common branch.

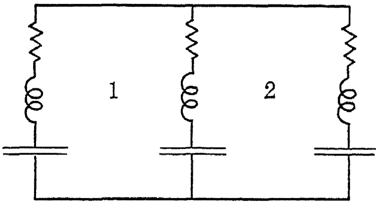

FIG. 66

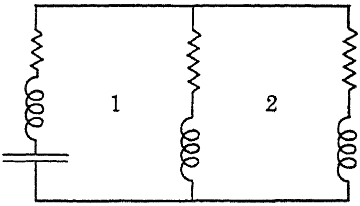

FIG. 67

Strictly speaking, this method of elimination is the rigorous procedure for integrating the given set. We chose the direct method in this chapter because it is much simpler. The only difficulty with the direct method arises when the determinantal equation comes out with negative powers of p, or appears to start with too low a positive power. This, however, can easily be rectified by simply multiplying it through afterward by the proper power of p. The question which we have to settle now is what this proper power is.

From what we have just been saying, this is quite simple to answer. Namely, the determinantal equation must be multiplied through by that power of p which equals the number of meshes which contain capacitance independently, i.e., in which charge may be independently specified. This is not always equal to that power which will just eliminate all the negative powers of p, otherwise we should not have spent all this time and space on such a simple matter.

To illustrate this point consider the two-mesh network of Fig. 68. For this the coefficients c_{ik} become:

$$\left.\begin{aligned} c_{11} &= L_1p + Sp^{-1} \\ c_{12} &= -\,Sp^{-1} \\ c_{22} &= L_2p + Sp^{-1} \end{aligned}\right\} \tag{394}$$

so that:

$$D(p) = c_{11}c_{22} - c_{12}^2 = L_1L_2p^2 + S(L_1 + L_2). \tag{395}$$

This polynomial is already free from negative powers of p, but the correct determinantal equation for this network requires the introduction of another power of p because we have one mesh in which charge may be independently specified. The correct equation is:

$$L_1L_2p^3 + S(L_1 + L_2)p = 0. \tag{396}$$

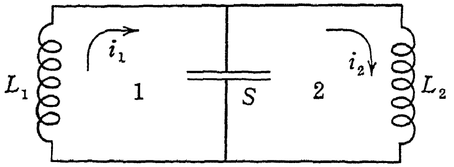

FIG. 68

It must be a cubic because we have three initial conditions to specify altogether, namely initial current and charge in mesh #1, and initial current in mesh #2. The system has three modes, which are the roots of (396).

These are:

$$\left.\begin{aligned} p_1 &= 0 \\ p_2 &= +j\sqrt{\frac{S(L_1 + L_2)}{L_1L_2}} \\ p_3 &= -j\sqrt{\frac{S(L_1 + L_2)}{L_1L_2}}\,. \end{aligned}\right\} \tag{397}$$

Thus we see that the system of Fig. 68 possesses one pair of conjugate imaginary modes, and one real mode whose value is zero. This zero mode may seem trivial, but fundamentally it is as important as the other two. The transient solutions take the form:

and:

$$\left.\begin{aligned} i_1 &= J_1^{(1)} + J_1^{(2)}e^{p_2t} + J_1^{(3)}e^{p_3t} \\ i_2 &= J_2^{(1)} + J_2^{(2)}e^{p_2t} + J_2^{(3)}e^{p_3t} \end{aligned}\right\} \tag{398}$$

thus giving rise to three integration constants according to what was said in the last section. If we had overlooked the zero mode, the solutions (398) would not have contained the constants $J_1^{(1)}$ and $J_2^{(1)}$, and consequently could not have been correct. In fact there would have been no way in which to meet the specified initial conditions of our problem.

It should be noted that this question of multiplying the polynomial $D(p)$ by the proper power of p before it correctly represents the determinantal equation when set equal to zero, agrees precisely with what has already been said regarding the degree of that equation and the determination of the number of modes of the given system by inspection of the network. All these points bear out the fundamental necessity for the agreement between the number of modes, the number of integration constants, and the number of initial conditions which may be specified. If the student will bear this simple correlation in mind, he will have no difficulty with any of the above matters pertaining thereto.

11. Illustrations and Approximations. In closing this chapter we shall illustrate the content of the above paragraphs by means of various special examples. Having mastered the fundamental principles governing the formation of the transient portion of the complete solution, it is a relatively simple matter to carry out the manipulations involved in a specific case.

Just as in the case of the steady-state solutions, it is not even necessary to set up the equilibrium equations. We have determined their solutions once and for all. We have done this for the most general case conceivable. It is merely necessary to substitute specific values into the various forms given above in order to carry out the solution to a specific problem.

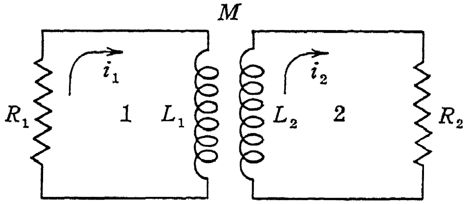

FIG. 69

Consider, for example, the network given by Fig. 69. This is a two-mesh affair with nothing but inductance and resistance and purely mutual coupling. No impressed force is shown because we wish to determine only the transient portion of the complete solution here. The system has two degrees of freedom. Two initial conditions may be specified, namely the two initial mesh-currents. Hence the system possesses two modes, and the determinantal equation will be of the second degree. It will not be necessary to multiply this equation by any power of p in order to give it its proper degree because neither mesh contains capacitance. All this we get by inspection. Hence we can write down the forms of the transient solutions in both meshes. These are:

and:

$$\left.\begin{aligned} i_1 &= J_1^{(1)}\epsilon^{p_1t} + J_1^{(2)}\epsilon^{p_2t} \\ i_2 &= J_2^{(1)}\epsilon^{p_1t} + J_2^{(2)}\epsilon^{p_2t} \end{aligned}\right\} \tag{399}$$

where p_1 and p_2 are the modes. In order to find these we must set up the determinant $D(p)$. By inspection we have:

$$\left.\begin{aligned} c_{11} &= L_1p + R_1 \\ c_{12} &= Mp = c_{21} \\ c_{22} &= L_2p + R_2. \end{aligned}\right\} \tag{400}$$

Hence the determinantal equation is:

$$(L_1p + R_1)(L_2p + R_2) - M^2p^2 = 0$$

or:

$$p^2 + \frac{L_1R_2 + L_2R_1}{L_1L_2 - M^2}p + \frac{R_1R_2}{L_1L_2 - M^2} = 0. \tag{401}$$

Solving in the usual way we have:

$$p = \frac{-(L_1R_2 + L_2R_1) \pm \sqrt{(L_1R_2 - L_2R_1)^2 + 4\,R_1R_2M^2}}{2\,(L_1L_2 - M^2)}. \tag{402}$$

The two modes p_1 and p_2 are obtained by using the plus and minus signs respectively in (402). It can be shown that the quantity under the radical is always positive so that both modes are real. It can further be shown that both modes are negative. This is hardly necessary since we know from physical considerations that they must be negative reals. Without elastance in the network there can be no oscillatory modes, and since the natural behavior must decay with time the modes must be negative.

It may be interesting to point out what form the modes take for the condition:

$$\frac{R_1}{L_1} = \frac{R_2}{L_2}. \tag{403}$$

This condition sometimes occurs in practice. It means that a coil having a given inductance also has a corresponding resistance. Then (402) becomes:

$$\left.\begin{aligned} p_1 &= \frac{-\sqrt{R_1R_2}}{\sqrt{L_1L_2}+M} \\ \text{and:}\qquad p_2 &= \frac{-\sqrt{R_1R_2}}{\sqrt{L_1L_2}-M}. \end{aligned}\right\} \tag{404}$$

The reader should note that M is always less than, or at most equal to $\sqrt{L_1L_2}$, so that p_2 is always negative. Also we see that the magnitude of p_2 is larger than that of p_1. The two become coincident for $M = 0$. This means that the two meshes have become isolated from each other.

Let us now see what we can gather as to the relative magnitudes of the normal functions involved. Here we have:

$$J_1^{(1)} : J_2^{(1)} = C_{11}^{(1)} : C_{12}^{(1)} = -c_{22}^{(1)} : c_{21}^{(1)} \tag{405}$$

and:

$$J_1^{(2)} : J_2^{(2)} = C_{11}^{(2)} : C_{12}^{(2)} = -c_{22}^{(2)} : c_{21}^{(2)}. \tag{406}$$

Substituting the values from (400) into (405) and (406), we get:

$$\frac{J_1^{(1)}}{J_2^{(1)}} = -\frac{L_2p_1 + R_2}{Mp_1} \tag{407}$$

and:

$$\frac{J_1^{(2)}}{J_2^{(2)}} = -\frac{L_2p_2 + R_2}{Mp_2}. \tag{408}$$

For the special case (403) these become:

$$\frac{J_1^{(1)}}{J_2^{(1)}} = \sqrt{\frac{R_2}{R_1}} = \sqrt{\frac{L_2}{L_1}} \backsimeq \frac{n_2}{n_1} \tag{407a}$$

and:

$$\frac{J_1^{(2)}}{J_2^{(2)}} = -\sqrt{\frac{R_2}{R_1}} = -\sqrt{\frac{L_2}{L_1}} \backsimeq -\frac{n_2}{n_1} \tag{408a}$$

where n_1 and n_2 are the turns in the two coils respectively. Hence, we see that for the special case (403), the amplitudes for the mode p_1 are roughly inversely proportional to the ratio of turns on the two coils. For the mode p_2 they are roughly inversely proportional to the negative ratio of turns.

Summarizing, we see that the natural behavior of the network of Fig. 69 is composed of a superposition of two exponentially decaying functions in both meshes. When the coupling is tight, the modes are widely different. For the limit $M \rightarrow \sqrt{L_1L_2}$, (402) evaluates to:

$$p_1 = -\frac{R_1R_2}{L_1R_2 + L_2R_1}$$

and:

$$p_2 = -\infty.$$

When the coupling is loose, the modes approach each other. For the limit $M \rightarrow 0$, (402) gives rise to:

$$p_1 = -\frac{R_1}{L_1}$$

and:

$$p_2 = -\frac{R_2}{L_2}.$$

The same general tendencies are true for the special condition (403), of course. For this case we can also roughly determine the relative magnitudes of the normal function amplitudes in the two meshes, and find that these are approximately inversely proportional to the ratio of turns, this ratio being positive for p_1 and negative for the mode p_2. The natural function corresponding to p_2 will in general not be as important as that for p_1, because the former will decay so much faster, especially when the leakage flux is small as it is in most transformers.

All this, of course, does not yet give us a complete picture of the net behavior of this network, because the steady state is still missing, and the magnitudes of the transient current amplitudes have not yet been evaluated. This latter operation will be taken up in the next chapter. We wish to point out, however, how much information can be obtained relative to the natural behavior of a network without carrying the analysis through to a finished result. The student should further notice how much more useful an analytic investigation, such as the above, is than a specific numerical one. It would have been much easier for us to demonstrate a numerical case with definite values for the resistances and inductances. But this would have told us nothing at all about how the modes vary with various degrees of coupling, or what happens when the special condition (403) is met. These things furnish interesting side-lights on the general behavior of such a network. They enlarge the horizon and enable us to predict new behaviors or remedy undesirable ones. Numerical examples are resorted to only when the problem gets so complicated that all other methods of analysis fail. They are also good for students to practice the routine on. In this respect the writer suggests that the above problem be worked out with several sets of numerical values which may be taken from laboratory data or elsewhere.

We continue now with another common type of network shown in Fig. 70. This is the same as the preceding except that we have now introduced elastance into the meshes in addition to the resistance and inductance. Again we have a system with two degrees of freedom, but now we can specify initial charges and currents in both meshes. Hence the system has four modes, and the determinantal equation will be of the fourth degree. The polynomial $D(p)$ will have to be multiplied by p^2 before setting it equal to zero in order to yield the determinantal equation. The transient solutions are of the form:

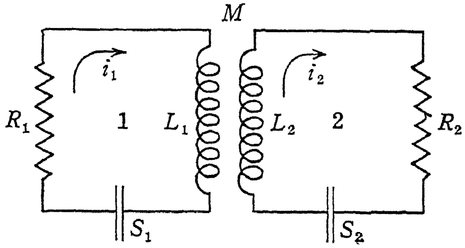

FIG. 70

$$\left.\begin{aligned} i_1 &= J_1^{(1)}e^{p_1t} + J_1^{(2)}e^{p_2t} + J_1^{(3)}e^{p_3t} + J_1^{(4)}e^{p_4t} \\ \text{and:}\quad & \\ i_2 &= J_2^{(1)}e^{p_1t} + J_2^{(2)}e^{p_2t} + J_2^{(3)}e^{p_3t} + J_2^{(4)}e^{p_4t}. \end{aligned}\right\} \tag{409}$$

If the resistances are small, the modes will all be complex and will come in two conjugate pairs. In that case the corresponding J's will also be conjugate pairs. We have for the elements of $D(p)$:

$$\left.\begin{aligned} c_{11} &= L_1p + R_1 + S_1p^{-1} \\ c_{12} &= c_{21} = Mp \\ c_{22} &= L_2p + R_2 + S_2p^{-1}. \end{aligned}\right\} \tag{410}$$

Hence the determinantal equation is:

$$\begin{aligned} (L_1L_2 - M^2)p^4 + (L_1R_2 + L_2R_1)p^3 + (R_1R_2 + L_1S_2 + L_2S_1)p^2 \\ + (R_1S_2 + R_2S_1)p + S_1S_2 = 0. \end{aligned} \tag{411}$$

This is a quartic, and in general presents quite a bit of difficulty in the process of determining its roots. In communication work, however, the networks of this type are usually highly oscillatory, i.e., the resistances are relatively small. In such cases, approximations may be applied, by means of which the roots of a determinantal equation such as (411) may be obtained with comparatively little effort and with sufficient accuracy for most purposes. This we wish to take up now.

The approximation method which we shall describe here is based fundamentally upon the convergence principle of an ordinary Taylor series. It is useful for obtaining the solutions to many different types of problems and is by no means confined to the particular situation above. This method, which is called the **Newtonian approximation method**, proceeds from the knowledge of an approximate solution, whatever form this may have, and arrives at a closer approximation by a series of steps which are fundamentally a repetition of the same process. The reader should master the philosophy of the method rather than follow the outline of its application to a specific case. In other words, the steps involved should have a definite meaning to him — they should appear to be the logical thing to do. Then the process becomes one which is easy to remember and just as easy to apply to new situations when they arise. It may be said in passing that approximation methods are equally or more important to the engineer than exact methods. The latter are very frequently too cumbersome, and in some cases impossible to carry out. Approximate methods are time saving and usually serve the purpose as satisfactorily as the exact ones.

Before the student can really grasp the true significance of Newton's approximation method, he must obtain a clear picture of what is meant by a Taylor expansion. Suppose we consider an arbitrary function of an independent variable which we may designate by p, and write for this function:

$$f(p).$$

Let us assume further that this function is represented graphically by the Fig. 71. In this case the function is evidently real because we have represented it completely in terms of real values. We shall proceed on this basis for the time being. With this graph given we can, of course, pick off the value of $f(p)$ for any specified value of p. For instance for $p = p_0$ the function is zero.

Now suppose we had no graph given, but instead knew that:

$$f(p_0) = 0 \tag{412}$$

and also knew the slope of the curve at the point $p = p_0$. This slope, of course, is given by:

$$\left(\frac{df}{dp}\right)_{p=p_0} = f'(p_0) \tag{413}$$

where the prime indicates the derivative. With this information alone, let the question be: What is the value of the function at the

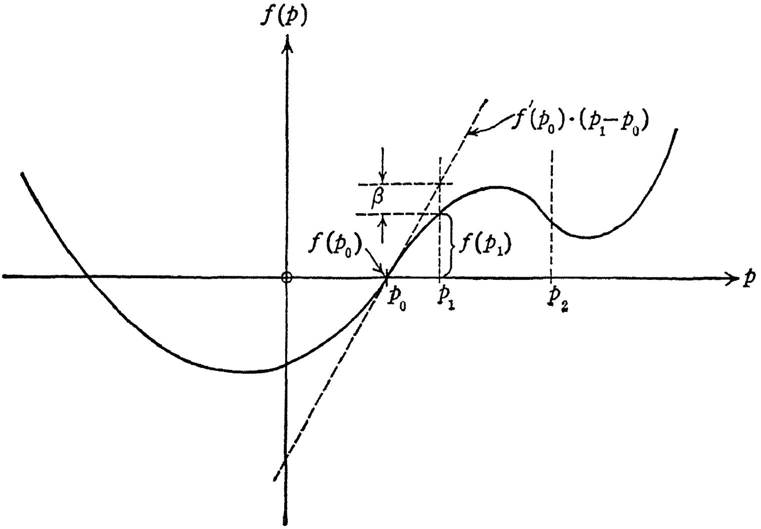

FIG. 71

point $p = p_1$, indicated in the Fig. 71? Obviously, with the given data, we could not answer this question exactly, but we could say that the answer would be approximately given by:

$$f(p_1) = f(p_0) + f'(p_0) \cdot (p_1 - p_0) + \cdots \tag{414}$$

where the plus sign and the dots at the end mean that this expression is only approximate as it stands, and that there is really more to come which at the present time we know nothing about. The eq. (414) says in effect that the value of $f(p)$ at $p = p_1$ is equal to the value at $p = p_0$ plus an increment which is approximately equal to the slope of $f(p)$ at $p = p_0$ times the horizontal distance $(p_1 - p_0)$. This is clear from the geometry of the figure. In the latter we also see that the discrepancy which exists between the result (414) and the correct value is indicated by β. The reason for the discrepancy is that the actual curve does not continue in a straight line after leaving the point $p = p_0$, but bends toward the right. If the point p_1 had been taken closer to the point p_0, then the discrepancy would have been smaller. On the other hand, if we had attempted to calculate the value of the function at the point p_2 in the above manner, we should have been very materially off the track. Obviously in that case it would become necessary to add another term to the expression (414) which would tend to correct the initial error.

The reader should recognize that by this method we are attempting to find the true value of a function at some point by starting with a known value at a neighboring point and adding or subtracting small increments to it in successive steps. We are approximating the true value by a succession of approximations or corrections to the value at the neighboring point. This value is represented by $f(p_0)$ in the expression (414). The next term is called the **first correction term.** If another term were added, it would be called the **second correction,** and so on. These successive terms all have a geometrical significance, similar to that for the first correction.

In treatises on function theory it is shown what the mathematical forms of the second, third, etc. corrections are. It turns out that these further corrections are successively proportional to the second, the third, etc. derivatives of the function at the point $p = p_0$. Just as the first correction is proportional to the slope of the function at p_0, so the second correction is proportional to the curvature of the function at p_0; the third correction is proportional to the rate of change of the curvature at p_0, etc. Written out completely in mathematical form, this process becomes:

$$f(p_1) = f(p_0) + \frac{1}{1!}f'(p_0) \cdot (p_1 - p_0) + \frac{1}{2!}f''(p_0) \cdot (p_1 - p_0)^2 + \frac{1}{3!}f'''(p_0) \cdot (p_1 - p_0)^3 + \cdots \qquad \textbf{(415)}$$

where the double-prime stands for the second derivative, the triple-prime for the third, etc. Note also that the second approximation is proportional to the square of the distance from p_0 to p_1, the third approximation to the cube of this distance, etc. We are giving no proof for this form (415), but are trying to point out that it is the logical form for geometrical reasons. In this respect note that the second correction itself represents a parabola with its apex at p_0. For our case illustrated by Fig. 71 this parabola would have to turn downward from p_0 so that when added to the straight line:

$$f'(p_0) \cdot (p_1 - p_0)$$

it would produce a resulting curve which bends toward the right like the correct one. By means of (415) we are virtually representing the actual curve $f(p)$ by means of a sum of curves, the first of which is a constant, the second a straight line, the third a quadratic parabola, the fourth a cubic parabola, etc. Thus we are approximating the given curve by means of a sum of elementary ones. Each added curve introduces a new tendency to bend, which brings about a closer agreement between the actual and the sum of elementary curves. This agreement is best in the neighborhood of p_0 and becomes less exact for points farther removed from this one. The agreement at any point becomes better as more terms are added to the expression (415).

This method of approximating a function by means of a sum of elementary functions is usually referred to as a Taylor expansion of $f(p)$, about the point, or in the neighborhood of the point $p = p_0$. A Taylor expansion is itself an approximation method. Hence, it is logical that approximation methods in general should have much in common with this kind of an expansion. We come now to the principle of Newton's method in particular.

We have seen geometrically how the actual curve of Fig. 71 may be successively approximated by means of a sum of elementary curves. There is another way in which this can be done which bears a very close relationship to this method. Suppose we return again to the expression (414) which contains only the function at p_0 and the first correction term. If we choose p_1 close to p_0, the value given by (414) will be almost correct. Having thus determined $f(p_1)$, we choose the point p_1 as an origin for a new expansion in the form of (414) and determine the value of the function a short distance removed from p_1. Then we move to this point and expand again to find $f(p)$ for a point still farther along. Thus by repeatedly applying only one correction term, we can eventually trace out the entire curve by successively moving our point of expansion. Graphically, we are approximating the actual curve by means of a succession of short tangents. The accuracy depends upon the shortness of the intervals at which new tangents are drawn. Each new tangent corresponds to a new expansion or a new application of the form (414). Thus by successively applying the same approximation over again, and at the same time shifting the origin of the expansion, we can determine a section of the given curve to within any degree of accuracy depending upon the number of applications. This is the essence of Newton's method of approximation. The essential formula for this method is given by (414), and it is very illuminating to recognize that this is nothing more than the first two terms of a Taylor expansion about the point $p = p_0$.

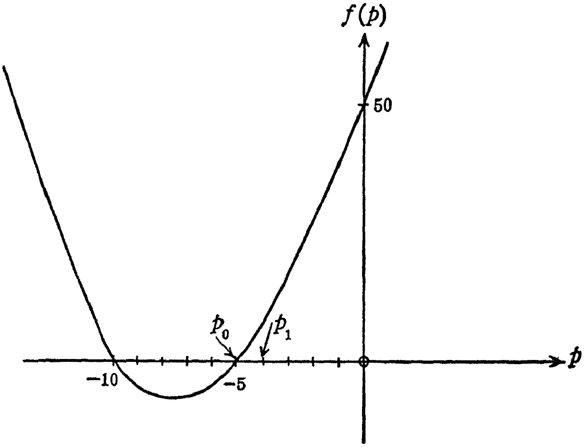

FIG. 72

Let us return now to the problem of finding the roots of an algebraic equation. Consider for simplicity the numerical quadratic equation:

$$f(p) = p^2 + 15\,p + 50 = 0. \qquad (416)$$

A plot of this function is given in Fig. 72. The intercepts of $f(p)$

with the p-axis are the desired roots. In applying the approximation method to this problem, we follow the inverse procedure to that outlined above. There we assumed that an interval in the independent variable, namely $(p_1 - p_0)$, was given, and we wished to determine the corresponding increment in the function $f(p)$. Now the function is given by (416), and our problem is to calculate the interval in p between an actual root, say $p = p_0$, and an approximate value for this root. Suppose the root we are after is the one $p_0 = -5$ in the figure. We do not know this value, but somehow we know that it is in that neighborhood, say -4. Let $p_1 = -4$. If now we can determine the interval $p_0 - p_1$, then obviously we can correct the value -4, and thus find the true root.

This idea is carried out in the following way. Consider the first two terms of a Taylor expansion of the function about the point $p = p_1$, thus:

$$f(p) = f(p_1) + f'(p_1)(p - p_1). \tag{417}$$

Now let:

$$p = p_0$$

so that:

$$(p - p_1) = (p_0 - p_1) = \delta \tag{418}$$

the desired correction. Then (417) becomes:

$$f(p_0) = 0 = f(p_1) + f'(p_1) \cdot \delta$$

and:

$$\delta = -\frac{f(p_1)}{f'(p_1)}. \tag{419}$$

Let us apply this to the equation (416). We have:

$$f(p_1) = (-4)^2 + 15\,(-4) + 50 = 6$$

$$f'(p) = 2\,p + 15 \qquad f'(p_1) = -8 + 15 = 7.$$

Hence:

$$\delta = -\tfrac{6}{7} = -0.857.$$

By (418) the desired root is given by:

$$p_0 = p_1 + \delta$$

so that we have in our case:

$$p_0 = -4 - 0.857 = -4.857$$

which is pretty near to the correct value. Now let:

$$p_1 = -4.857.$$

Then:

$$f(p_1) = 0.765 \qquad f'(p_1) = 5.286.$$

Hence:

$$\delta = -\frac{0.765}{5.286} = -0.144.$$

This is the second correction, so that we have:

$$p_0 = -4.857 - 0.144 = -5.001$$

which is correct to within one-tenth of one per cent. Any degree of accuracy can be obtained by further applying the same method.

It is not even necessary to know an approximate value to one of the roots. The method will automatically converge toward the next nearest root from any starting point. Take the eq. (416) again and assume that:

$$p_0 = 0 + \delta$$

i.e., that:

$$p_1 = 0.$$

This gives:

$$f(p_1) = 50 \qquad f'(p_1) = 15$$

and hence:

$$\delta = p_0 = -\tfrac{50}{15} = -3.33.$$

Two more applications will suffice to obtain the first root with sufficient accuracy.

The method is always applicable in the same form no matter what the degree of the equation is, or whether the roots are real or complex. A Taylor series holds for complex arguments as well as for real ones. Hence, Newton's approximation method operates for complex values as well. The only requirement in a complex case is that the first approximation be close enough to the actual value of the root to make the process converge. When a root is found, it is best to divide it out and treat the remainder as an original equation, or if the remainder is of the second degree, the quadratic formula may be used. In the case of complex roots, the process can not be illustrated graphically as simply as was done above, but this does not make the process any harder to understand fundamentally.

With this short discourse on the approximation method, let us return to our determinantal equation (411) for the network given in Fig. 70. If the network resistances are high so that no oscillations are expected, then the roots of (411) will all be real. Their determination needs no further comment. However, when the network is oscillatory or highly oscillatory, as is usually the case in communication work, then a slightly different procedure will enable us to arrive at the roots more rapidly.

In such a case we first evaluate the roots to that equation which is obtained from (411) by neglecting the resistances altogether. This is:

$$(L_1L_2 - M^2)p^4 + (L_1S_2 + L_2S_1)p^2 + S_1S_2 = 0. \qquad (420)$$

Notice that this might have been obtained from (411) by simply cancelling all the odd powered terms, except for the portion R_1R_2 of the square term which is very small compared to the rest of this coefficient. Eq. (420) is a bi-quadratic. Considering p^2 as the variable, it may be solved by the quadratic formula. These values of p are then used together with the original eq. (411) in the determination of the actual roots by means of the approximation method. Thus the solutions of the dissipationless eq. (420) serve as the first approximations for the Newtonian method as applied to (411). The latter then becomes the $f(p)$ in (419). Note also in this connection that, since the approximate root p_1 satisfies (420), it will make the first, third, and fifth terms in (411) vanish, so that $f(p_1)$ retains only the second and fourth terms. This materially simplifies the numerical work.

In order to illustrate this, let us assign values to the parameters of the network of Fig. 70 and calculate the modes. Suppose:

$$\begin{array}{lll} L_1 = 0.001 & S_2 = 1.5 \cdot 10^9 & M = 0.001 \\ L_2 = 0.003 & R_1 = 5 & \\ S_1 = 10^9 & R_2 = 15 & \end{array}$$

where inductance is given in henries, resistance in ohms, and elastance in darafs. The eq. (420) for neglected resistance therefore becomes:

$$2 \cdot 10^{-6}p^4 + 4.5 \cdot 10^6p^2 + 1.5 \cdot 10^{18} = 0 \qquad (421)$$

which is equivalent to:

$$p^4 + 2.25 \cdot 10^{12}p^2 + 0.75 \cdot 10^{24} = 0. \qquad (421a)$$

Solving this, we get:

$$p^2 = -1.842 \cdot 10^{12} \qquad p^2 = -0.408 \cdot 10^{12} \tag{422}$$

from which:

$$p = \pm j\,1.355 \cdot 10^6 \qquad p = \pm j\,0.638 \cdot 10^6. \tag{423}$$

Note that these modes for neglected resistance are pure imaginaries in two pairs of conjugate values. This, we should expect of course. The modes will always be pure imaginaries for neglected resistance. Hence the p^2-roots of the equation for neglected resistance will always be negative reals.

The four values indicated by (423) are now to be used as the first approximations in applying the Newtonian method. When the network is highly oscillatory, as it is in this case, the values (423) are very close to the true values already. The major error so far is that they possess no negative real parts, i.e., no decrements. The Newtonian method will supply these. In order to be able to substitute into eq. (419), we must form $f(p)$ and $f'(p)$. For this purpose we write down the complete determinantal eq. (411). With the above numerical values, this becomes:

$$\begin{aligned} f(p) = 2 \cdot 10^{-6} p^4 + 0.03\, p^3 + 4.5 \cdot 10^6\, p^2 \\ + 2.25 \cdot 10^{10}\, p + 1.5 \cdot 10^{24}. \end{aligned} \tag{424}$$

The derivative of this function is:

$$f'(p) = 8 \cdot 10^{-6} p^3 + 0.09\, p^2 + 9 \cdot 10^6\, p + 2.25 \cdot 10^{10}. \tag{425}$$

Let us consider the first pair of values (423). Therefore we put:

$$p_1 = +j\,1.355 \cdot 10^6. \tag{423a}$$

It is not necessary to consider the conjugate of this also, since it will give rise to the same result, as the reader may see for himself. Using (423*a*) in (424) and (425), we get:

$$\text{and:} \quad \left.\begin{aligned} f(p_1) &= -j\,4.42 \cdot 10^{16} \\ f'(p_1) &= -j\,7.77 \cdot 10^{12}. \end{aligned}\right\} \tag{426}$$

It may seem to the reader that this substitution involves quite a bit of numerical work, but this is not the case. In forming $f(p_1)$, the even powered terms need not be considered because these add to zero anyway because (423*a*) is a root of (421*a*). In forming $f'(p_1)$, on the other hand, the even powered terms do not add to zero, but they are relatively so small that they can be neglected. The reader should carry out these substitutions in order to clarify his mind on these points. Finally, using (426) in (419) we get:

$$\delta = -5.7 \cdot 10^3 \tag{427}$$

This is the decrement that goes with the first pair of imaginaries of (423).

The decrement for the second pair of imaginaries of (423) is found in the same way. Namely, let:

$$p_1 = +j\,0.638 \cdot 10^6. \tag{423b}$$

Then, substituting into (424) and (425), and noting the same simplifications as above, we have:

and:

$$\left.\begin{aligned} f(p_1) &= +j\,6.5 \cdot 10^{15} \\ f'(p_1) &= +j\,3.66 \cdot 10^{12}. \end{aligned}\right\} \tag{428}$$

Using these values in (419) again, we have:

$$\delta = -1.78 \cdot 10^3 \tag{429}$$

which is the required decrement. The four modes are therefore:

$$\left.\begin{matrix} (-5.7 + j\,1355) \cdot 10^3 & (-1.78 + j\,638) \cdot 10^3 \\ (-5.7 - j\,1355) \cdot 10^3 & (-1.78 - j\,638) \cdot 10^3. \end{matrix}\right\} \tag{430}$$

The transient solutions (409) therefore become:

$$\left.\begin{aligned} i_1 &= e^{-5700t}(J_1{}^{(1)}e^{j\,1.355 \cdot 10^6 t} + J_1{}^{(2)}e^{-j\,1.355 \cdot 10^6 t}) \\ &\quad + e^{-1780t}(J_1{}^{(3)}e^{j\,.638 \cdot 10^6 t} + J_1{}^{(4)}e^{-j\,.638 \cdot 10^6 t}) \\ i_2 &= e^{-5700t}(J_2{}^{(1)}e^{j\,1.355 \cdot 10^6 t} + J_2{}^{(2)}e^{-j\,1.355 \cdot 10^6 t}) \\ &\quad + e^{-1780t}(J_2{}^{(3)}e^{j\,.638 \cdot 10^6 t} + J_2{}^{(4)}e^{-j\,.638 \cdot 10^6 t}). \end{aligned}\right\} \tag{431}$$

Each pair of terms represents a damped oscillation. As has been pointed out, the J's are complex, and hence contain the phase angles of the respective terms. The following amplitudes are in the form of conjugate pairs:

$$\begin{matrix} (J_1{}^{(1)} \quad J_1{}^{(2)}) & (J_1{}^{(3)} \quad J_1{}^{(4)}) \\ (J_2{}^{(1)} \quad J_2{}^{(2)}) & (J_2{}^{(3)} \quad J_2{}^{(4)}). \end{matrix}$$

Each pair defines one real amplitude and one real phase.

There remains one more thing to point out in connection with the approximate method for evaluating the roots of the determinantal equation. In the above example the degree of this equation was even. When it is odd, the procedure is slightly different. In this case the equation contains one negative real root even though the rest be conjugate complexes. This real root should be evaluated first by applying Newton's method directly. The first approximation may frequently be assumed zero as was shown in the case of eq. (416) above. After this real root is found, it should be divided out, and the remainder treated as in the case where the degree is even.

These illustrations should suffice to show how in general the transient portion of the complete network solution is set up, and what the various parts mean. Very often a knowledge of the form of the solution, i.e., the number of modes, decrements, frequencies, and relative amplitudes of the normal functions, suffices to throw the required amount of light upon the particular situation being treated. A knowledge of these things quite definitely outlines the essentials of the network's natural behavior. However, the complete answer is not found until the amplitudes of the transient terms have been definitely determined in agreement with specified initial conditions. This, we shall now take up, after which we shall be in a position to give more thorough attention to the matter of illustrations, which so far could be only fragmentary in nature.

PROBLEMS TO CHAPTER V

5-1. How many modes does the network of Prob. **4-4** have? How many initial conditions may be specified? What are these? Set up the determinantal equation. Set up the form of the transient solution, and show that the number of integration constants agrees with the number of initial conditions.

5-2. Set up the determinantal equation for the network of Fig. 59. Write down the form of the transient solution. Add to this the solution of Prob. **4-2** to form the complete solution to this network. From this current solution determine the mesh-charge solutions.

5-3. How many modes does the network of Prob. **4-3** have? What is the form of the transient current solution? What seems to be wrong with this situation? Discuss from a physical standpoint.

5-4. How many modes does the network of Fig. 57 have? What are these? What is the form of the transient current solution in this case? Show that the number of integration constants agrees with the number of initial conditions that may be specified.

5-5. Set up the differential-integral operators for the network given in Fig. 65, assigning arbitrary values to the parameters. In terms of these set up the instantaneous equilibrium conditions. Write down the form of the transient solution in the various meshes: (*a*) In terms of the transient current amplitudes; (*b*) in terms of the minors of the modular determinant and the true integration constants. Specify the initial conditions for the evaluation of these constants. Determine the form of the corresponding transient mesh-charges.

CHAPTER VI

THE EVALUATION OF INTEGRATION CONSTANTS

1. The Direct Method. There are various ways in which the evaluation of the integration constants may be carried out. They all lead to the same result in the end, of course. It is more or less a matter of taste as to which method should be used in a particular case. In general the amount of labor involved is about the same, no matter how the problem is attacked, and it may be stated at the outset that it is quite considerable. The matter of outlining the process is, however, quite simple and direct, and involves nothing essentially new. It merely becomes cumbersome in the matter of notation and the space occupied by the equations themselves. This is an inherent characteristic of network analysis, and the reader will probably have gotten somewhat used to the formidable appearance of the solutions by this time. In order to reduce the bulk to some extent, we may make use of more compact methods of notation.

The evaluation of integration constants must proceed from a simultaneous consideration of the complete mesh-current solutions, and the initial conditions of the network. Suppose for this demonstration that we consider an impressed voltage in the first mesh only. The reader should be familiar enough with the general solution to be able to alter the special form which this assumption introduces, in order to suit other cases which may arise. The matter of where the voltage is impressed, or how many voltages are impressed simultaneously, affects the steady-state solution alone. The transient portion of the complete solution is a force-free expression of the network, and its form can therefore not be influenced in any way by the character or point of application of the driving force.

From Chapter IV we recall that the steady-state current in any mesh is given by:

$$(i_k)_s = I_k e^{j\omega t} \tag{432}$$

with an impressed voltage:

$$e_1 = E_1 e^{j\omega t} \tag{433}$$

in the first mesh, where:

$$I_k = \frac{E_1 B_{1k}}{D}. \tag{434}$$

In eq. (432) the parenthesis and subscript s on the current are used in order to make it evident that this is the steady-state solution alone. In both of the equations (432) and (433) the real portion of the right-hand side is, of course, understood as usual. Putting (432) and (434) together, we have:

$$(i_k)_s = \frac{E_1 B_{1k}}{D} e^{j\omega t}. \tag{435}$$

Both B_{1k} and D are functions of $j\omega$, of course.

To this steady-state portion must now be added the corresponding transient solution given by the equations (363) of Chapter V. For mesh #k this gives:

$$(i_k)_t = J_k^{(1)} e^{p_1 t} + J_k^{(2)} e^{p_2 t} + \cdot \cdot \cdot + J_k^{(2n)} e^{p_{2n} t} \tag{436}$$

where the subscript t on the left is used to distinguish this as the transient solution. Here we now introduce the more compact notation:

$$(i_k)_t = \sum_{\nu=1}^{2n} J_k^{(\nu)} e^{p_\nu t}. \tag{436a}$$

It is expedient to write this solution in terms of the quantities $G^{(\nu)}$ which we introduced in the last chapter. It will be remembered that a knowledge of these constants enabled us to determine any normal function amplitude. Equation (378) of Chapter V expresses any of the latter in terms of one of these constants and a corresponding minor of an element in the first row of $D(p)$. Using this in (436a), we have:

$$(i_k)_t = \sum_{\nu=1}^{2n} C_{1k}^{(\nu)} G^{(\nu)} e^{p_\nu t}. \tag{437}$$

The complete solution for any mesh-current is now given by the linear sum of (435) and (437). We include the $\mathcal{R}e$-operator for completeness, thus:

$$i_k = \mathcal{R}e\left[\frac{E_1 B_{1k}}{D} e^{j\omega t}\right] + \sum_{\nu=1}^{2n} C_{1k}^{(\nu)} G^{(\nu)} e^{p_\nu t}. \tag{438}$$

Note that no $\mathcal{R}_e$-operator is required for the transient portion. The latter comes out real of its own accord as explained in the last chapter.

The complete solution (438) contains $2n$ arbitrary constants $G^{(\nu)}$ for the general case where each mesh in the network contains independent inductance and capacitance. In this case we can specify n initial currents and n initial charges. Hence, the constants can be uniquely evaluated. The initial currents may be substituted into the solution (438) directly. The initial charges cannot be substituted directly into this result. We must first form the expression for the mesh charge. This, it will be recalled, is simply:

$$q_k = \int_0^t i_k \, dt. \tag{439}$$

Substituting (438) into (439) and carrying out the integration, we have:

$$q_k = \mathcal{R}_e\left[\frac{E_1B_{1k}}{j\omega D}e^{j\omega t}\right] + \sum_{\nu=1}^{2n} C_{1k}{}^{(\nu)}G^{(\nu)}p_\nu{}^{-1}e^{p_\nu t}. \tag{440}$$

We are now ready to substitute the initial conditions. Suppose we denote the initial charges by q_{k0} and the initial currents by i_{k0}. Then we have by (440) and (438):

$$\left.\begin{aligned} q_{k0} &= \mathcal{R}_e\left[\frac{E_1B_{1k}}{j\omega D}\right] + \sum_{\nu=1}^{2n} C_{1k}{}^{(\nu)}G^{(\nu)}p_\nu{}^{-1} \\ \text{and:}& \\ i_{k0} &= \mathcal{R}_e\left[\frac{E_1B_{1k}}{D}\right] + \sum_{\nu=1}^{2n} C_{1k}{}^{(\nu)}G^{(\nu)} \end{aligned}\right\} (k = 1, 2, \ldots n). \tag{441}$$

By putting the subscript k successively equal to 1, 2, . . . n, as indicated, we obtain n charge equations and n current equations, thus making a total of $2n$ equations which just suffices for the evaluation of the $2n$ constants $G^{(1)}, G^{(2)}, \ldots G^{(2n)}$. When these are known, all the normal function amplitudes are known from eq. (378) of Chapter V; or the complete solutions may be written in terms of these constants directly as in (438) above.

The important point to note is that this method of evaluation makes it clear that the $2n$ initial conditions just suffice to evaluate the corresponding number of degrees of flexibility in the complete solution. The result (441) is extremely compact. Its interpretation requires a thorough knowledge of the notation and method of expression used for the representation of the steady state and transient portions as discussed in the previous chapters. There is nothing haphazard about obtaining the complete network solution in this fashion. It is direct, and incurs no lost motion. It organizes the work to the highest degree, and at the same time makes it as brief as possible. In spite of this the amount of actual labor involved is considerable when a large number of meshes are involved. We will give illustrations later which bear out this point.

2. Evaluation by Means of Successive Initial Derivatives. We now take up an alternative method of evaluating the normal function amplitudes in the transient solution (436*a*), which may in some cases be a little faster than the above method. It lacks, however, the straightforwardness of that method, and furthermore does not make evident the agreement between the number of arbitrary constants and the corresponding number of initial conditions. On the other hand, it gives us a new viewpoint on the network solution in general which may prove useful in some cases. We proceed now with this method.

Expressing the complete solution as a sum of (435) and (436*a*), we have:

$$i_k = \mathcal{R}_e\left[\frac{E_1 B_{1k}}{D} e^{j\omega t}\right] + \sum_{\nu=1}^{2n} J_k^{(\nu)} e^{p_\nu t}. \qquad (442)$$

For any mesh-current i_k this involves $2n$ unknown amplitudes $J_k^{(\nu)}$. Hence $2n$ specific equations involving these quantities are in general necessary for their evaluation. These may be obtained in the following way.

The expression (442) may be differentiated with respect to time any finite number of times. If we differentiate $2n - 1$ successive times, and each time put $t = 0$, we will get the following system of $2n$ equations:

$$\left.\begin{aligned}
i_{k0} &= \Re e\left[\frac{E_1 B_{1k}}{D}\right] + \sum_{\nu=1}^{2n} J_k^{(\nu)} \\
\left(\frac{di_k}{dt}\right)_0 &= \Re e\left[\frac{E_1 B_{1k}}{D} j\omega\right] + \sum_{\nu=1}^{2n} J_k^{(\nu)} p_\nu \\
\left(\frac{d^2 i_k}{dt^2}\right)_0 &= \Re e\left[\frac{E_1 B_{1k}}{D} (j\omega)^2\right] + \sum_{\nu=1}^{2n} J_k^{(\nu)} p_\nu^2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\left(\frac{d^{2n-1} i_k}{dt^{2n-1}}\right)_0 &= \Re e\left[\frac{E_1 B_{1k}}{D} (j\omega)^{2n-1}\right] + \sum_{\nu=1}^{2n} J_k^{(\nu)} p_\nu^{2n-1}.
\end{aligned}\right\} \tag{443}$$

This system (443) could be solved simultaneously for the $2n$ amplitudes $J_k^{(1)}, J_k^{(2)}, \ldots J_k^{(2n)}$, provided the left-hand members were known. These are successively, the initial current in the kth mesh, the first derivative of that current at $t = 0$, the second derivative at $t = 0$, etc., down to the $(2n - 1)$th derivative at $t = 0$. We call these members the **successive initial derivatives** of the current in the kth mesh.

The method of evaluation indicated by (443) must be carried out separately for each mesh-current. It is perfectly general and can always be done. It presupposes a knowledge of the successive initial derivatives, however. We shall now discuss separately how these may in general be determined from the specified initial conditions.

3. Determination of the Successive Initial Derivatives. For this purpose we must go back to the original inhomogeneous system of differential equations expressing the dynamic equilibrium of the forced network. This is given by the expression (303) of Chapter IV. We shall rewrite this system here in the condensed notation:

$$\sum_{k=1}^{n} a_{sk} i_k = e_s \qquad (s = 1, 2, \ldots n) \tag{444}$$

where for our case:

$$e_s \begin{Bmatrix} = e_1 & \text{for} & s = 1 \\ = 0 & \text{for} & s \neq 1 \end{Bmatrix}. \tag{445}$$

We recall that:

$$a_{sk} = L_{sk}\frac{d}{dt} + R_{sk} + S_{sk}\int dt. \tag{446}$$

Hence the system (444) may be written:

$$\sum_{k=1}^{n} L_{sk}\frac{di_k}{dt} = e_s - \sum_{k=1}^{n}\left(R_{sk} + S_{sk}\int dt\right) i_k. \tag{447}$$

If we let $t = 0$ in the above, we have:

$$\sum_{k=1}^{n} L_{sk}\left(\frac{di_k}{dt}\right)_0 = e_{s0} - \sum_{k=1}^{n}(R_{sk}i_{k0} + S_{sk}q_{k0}). \tag{448}$$

The right-hand members of this system of equations involve the initial mesh-currents and charges and the initial value of the driving force, all of which are known. The left-hand members involve the first initial mesh-current derivatives, which are unknown. The system (448) may therefore be solved for these first initial derivatives.

Now consider the derivative of the expression (447), which is:

$$\sum_{k=1}^{n} L_{sk}\frac{d^2i_k}{dt^2} = \frac{de_s}{dt} - \sum_{k=1}^{n}\left(R_{sk}\frac{d}{dt} + S_{sk}\right)i_k. \tag{449}$$

For $t = 0$ this becomes:

$$\sum_{k=1}^{n} L_{sk}\left(\frac{d^2i_k}{dt^2}\right)_0 = \left(\frac{de_s}{dt}\right)_0 - \sum_{k=1}^{n}\left\{R_{sk}\left(\frac{di_k}{dt}\right)_0 + S_{sk}i_{k0}\right\}. \tag{450}$$

Again the right-hand members are completely known. The left-hand members now involve the second initial derivatives, which may be solved for by the usual process of treating simultaneous algebraic equations.

By differentiating (447) twice and putting $t = 0$, we are able to determine the third initial derivatives. In the same manner any number of successive initial derivatives may be determined. The above may be summarized in the following way. If we denote the rth initial derivative of i_k by:

$$u_k^r$$

and the rth initial derivative of e_s by:

$$e_s^r$$

then we have by the above reasoning that:

$$\sum_{k=1}^{n} L_{sk}u_k^r + \sum_{k=1}^{n}(R_{sk}u_k^{r-1} + S_{sk}u_k^{r-2}) = e_s^{r-1} \tag{451}$$

$$(r = 1, 2, \ldots n.)$$

This formula makes it apparent that if we have two successive values of u_k^r to start with, say:

$$u_k^{-1} = q_{k0}$$

and:

$$u_k^0 = i_{k0} \tag{451a}$$

then any number of additional successive derivatives may be found. Such a formula as (451) is called a **recursion formula.** We meet it very frequently in analytic work, and shall have use for it again later on.

4. The Solution in the Form of a Taylor Expansion. In the above we have found a very interesting result. Namely, we have seen that if we know what the initial values of the integrals of the mesh-currents are and the initial mesh-currents themselves, then we can determine from these any number of successive initial derivatives. These are completely determined by the initial charges and currents alone. This is quite a remarkable result. It will probably be better appreciated when translated into its mechanical analog.

Suppose we consider a projectile shot from the mouth of a cannon. The initial position and velocity of the bullet correspond to the initial charge and current in an equivalent electrical system. Knowing these two, completely determines the initial acceleration, the initial rate of acceleration, and all successive initial derivatives thereof, as far as we wish to go. This is a fundamental principle in mechanics; namely, if we know the initial positions and velocities of a system of mass-points, together with the necessary parameters of the system of course, then all the successive initial derivatives of the velocity or position of the points are fixed. The result (451) translates this into the language of the electrical network.

In order to appreciate what this really means we must consider the following. In the preceding chapter we gave briefly the significance of a Taylor expansion for any function of a single variable. We saw there that a function could be completely specified in terms of its initial value, its initial derivative, and successive initial derivatives. Thus, for example, consider the x-component of the motion of a given mass-point, and let:

$$x = f(t). \tag{452}$$

This may be expanded in a Taylor series about the point $t = 0$, thus:

$$f(t) = f(0) + \frac{1}{1!}f'(0) \cdot t + \frac{1}{2!}f''(0) \cdot t^2 + \frac{1}{3!}f'''(0) \cdot t^3 + \cdots. \tag{453}$$

Substituting (452) into (453) gives:

$$x = x_0 + \frac{1}{1!}\left(\frac{dx}{dt}\right)_0 \cdot t + \frac{1}{2!}\left(\frac{d^2x}{dt^2}\right)_0 \cdot t^2 + \frac{1}{3!}\left(\frac{d^3x}{dt^3}\right)_0 \cdot t^3 + \cdot\cdot\cdot. \tag{454}$$

The same may be done with the y and z components of the motion. Hence, it is clear that the motion is completely determined by its successive initial derivatives.

In exactly the same way we could write for the current in any mesh of a network:

$$i_k = f(t) \tag{455}$$

and then from the Taylor expansion (453) obtain:

$$i_k = i_{k0} + \left(\frac{di_k}{dt}\right)_0 \cdot t + \frac{1}{2!}\left(\frac{d^2i_k}{dt^2}\right)_0 \cdot t^2 + \frac{1}{3!}\left(\frac{d^3i_k}{dt^3}\right)_0 \cdot t^3 + \cdot\cdot\cdot. \tag{456}$$

Obtaining the successive initial derivatives from (451) and substituting them into (456), gives us the complete solution. Note that this form for the solution does not set apart the steady-state and transient portions. The entire situation is lumped together in the one expansion. In the notation of eq. (451), we would have:

$$i_k = \sum_{r=0}^{\infty} u_k^r \cdot \frac{t^r}{r!}. \tag{457}$$

The utility of this form for the solution is not very great. So long as we wish to know the current only for very small values of time, very few terms in the expansion are necessary. However, if we wish to use this expansion to calculate the current after an appreciable elapse of time, then we need to consider a large number of terms. This requires the determination of a large number of initial derivatives, which might be more laborious than the determination of the complete solution in the regular manner.

The Taylor expansion for the current is interesting, however, in that it makes evident the fact that the current is completely specified by the initial charge and initial current. This, of course, agrees perfectly with what has been said regarding the solution in normal functions and the agreement between the arbitrary constants of integration and the initial conditions, i.e., their unique determination by means of the initial charges and currents.

The solution in normal functions can be shown to be equivalent to the Taylor expansion (456) or (457). Consider the form (442), for example. If we expand the exponential functions in a Taylor series, we get:

$$i_k = \sum_{r=0}^{\infty}\left\{\mathcal{R}_e\left(\frac{E_1B_{1k}(j\omega)^r}{D}\right) + \sum_{\nu=1}^{2n} J_k^{(\nu)}p_\nu^r\right\}\frac{t^r}{r!}. \tag{458}$$

Comparing (456) and (458), we see that:

$$\left(\frac{d^ri_k}{dt^r}\right)_0 = \mathcal{R}_e\left[\frac{E_1B_{1k}}{D}(j\omega)^r\right] + \sum_{\nu=1}^{2n} J_k^{(\nu)}p_\nu^r \tag{459}$$

which agrees with the set of equations (443) for the determination of the amplitudes $J_k^{(\nu)}$. Thus the method of evaluation of integration constants by means of the successive initial derivatives is fundamentally a process of expanding the solution in normal functions, in a Taylor series, and then equating the coefficients of like powers of t in this series and that for the direct expansion of the current. This is summarized by the expression (459). In this process, only the first $2n$ coefficients of the expansion need be considered in order to determine the unknown amplitudes $J_k^{(\nu)}$.

5. Comparison of the Normal Function Solution and the Taylor Expansion from the Practical Standpoint. The solution in normal functions consists essentially of the following steps: First, the determination of the steady-state solution; second, the determination of the form of the transient solution; third, the determination of the modes of the system, i.e., the solution of the determinantal equation; and fourth, the evaluation of the constants of integration. Of these, the last two are perhaps the most difficult from the standpoint of the amount of actual labor involved. The fourth step necessitates the determination of $2n$ initial derivatives in the general case, unless the direct method of evaluation is used. The solution in terms of a Taylor series on the other hand, requires only one step fundamentally, and that is the determination of a sufficient number of successive initial derivatives. Here the word "sufficient" is the factor which usually prohibits the use of the latter type of solution.

It may very well be, however, that we are interested in the solution only for a very short duration of time. If this time is so small that the Taylor expansion converges well enough with a reasonable number of terms, then the Taylor expansion will give the speedier solution, because the total amount of labor involved in getting the desired result will be considerably less by the latter method.

In some rather simple cases the Taylor expansion leads to the desired result very quickly. Let us show this by means of a few illustrations. Suppose we consider the sudden application of a constant voltage to an inductance of L henries and negligible resistance. The equation of dynamic equilibrium for this case is:

$$L\frac{di}{dt} = E. \tag{460}$$

Hence:

$$\left(\frac{di}{dt}\right)_0 = \frac{E}{L}. \tag{461}$$

All the successive derivatives of (460) are zero on account of the constant driving force. Hence, the Taylor expansion is limited to two terms. We have:

$$i = i_0 + \frac{E}{L}t \tag{462}$$

which is the correct solution as may be checked by ordinary integration of (460).

Again, consider an R, L, circuit with a suddenly applied constant voltage E. For this:

$$L\frac{di}{dt} + Ri = E. \tag{463}$$

For $t = 0$ this gives:

$$\left(\frac{di}{dt}\right)_0 = \frac{E}{L} - \frac{R}{L}i_0. \tag{464}$$

The recursion formula for successive initial derivatives becomes:

$$\left(\frac{d^ri}{dt^r}\right)_0 = -\frac{R}{L}\left(\frac{d^{r-1}i}{dt^{r-1}}\right)_0 \quad r > 1. \tag{465}$$

Hence, the Taylor expansion may be written down by inspection. It is:

$$i = i_0 + \left(\frac{E}{L} - \frac{R}{L}i_0\right)\left(t - \frac{1}{2!}\frac{R}{L}t^2 + \frac{1}{3!}\frac{R^2}{L^2}t^3 - \cdots\right). \tag{466}$$

Any number of terms may be added since the manner in which they are formed is clear. With a little manipulation we may put this in the form:

$$i = \frac{E}{R}\left[\frac{Rt}{L} - \frac{1}{2!}\left(\frac{Rt}{L}\right)^2 + \frac{1}{3!}\left(\frac{Rt}{L}\right)^3 - \cdots\right]$$
$$+ i_0\left[1 - \frac{Rt}{L} + \frac{1}{2!}\left(\frac{Rt}{L}\right)^2 - \frac{1}{3!}\left(\frac{Rt}{L}\right)^3 + \cdots\right]. \qquad (467)$$

But this we recognize to be:

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) + i_0 e^{-\frac{Rt}{L}} \qquad (467a)$$

which agrees with the solution in normal functions.

If the system is more complicated, then it is generally impossible to recognize the normal functions by inspection of the Taylor expansion. When the function is monotonic, such as an exponential rise or decay, or nearly monotonic, then the expansion will ordinarily converge very poorly, and cannot be used for purposes of calculation. If the function is known to be oscillatory, then relatively few terms will suffice to determine the initial character of the transient state. This may be appreciated from the following.

Consider the Taylor expansion of the sine function:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \qquad (468)$$

In Fig. 73 are plotted the various component curves, representing the individual terms in this expansion, and their sum for four terms. From this illustration we see that with four terms, the first half-cycle is fairly well represented.

Hence, the number of terms necessary in the Taylor expansion for a mesh-current depends upon the number of oscillations we wish to determine. It is quite common that we are interested merely in the first maximum of the current during the transient period. This may be the case in computing the behavior of a relay-operated device or in checking the approximate maximum transient in a circuit, for protective reasons. Then only relatively few terms in the Taylor expansion suffice to give the required answer approximately. If the system is at all complicated, it would be foolish to determine a complete solution in normal functions for this purpose. It takes experience and familiarity, of course, to determine which method is best in some cases. At all events the first $2n - 1$ initial derivatives may be calculated anyway, since these will be needed for the evaluation of the integration constants in the normal function solution. If the Taylor expansion with these $2n$ terms already suffices to give the desired result, then the calculations may be terminated there. If it does not, then either more successive derivatives may be found or else the complete solution carried through, according to what seems best. The preliminary investigation by means of $2n$ terms of the Taylor expansion is not a waste of time in either case, and may enable a decision as to whether the exact solution is necessary or not.

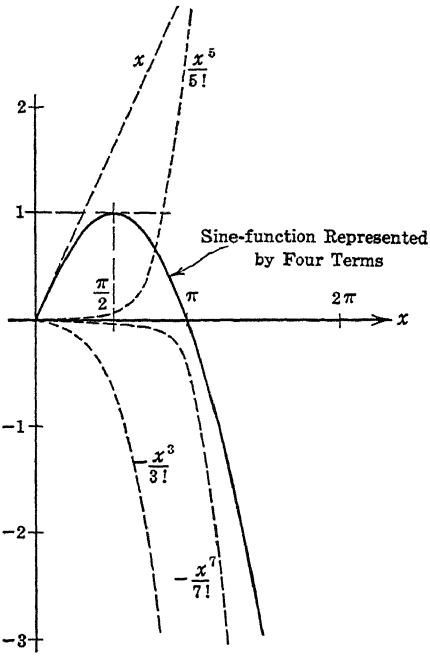

FIG. 73

6. Illustrations. Consider the two-mesh network shown in Fig. 74. Both meshes have a weight of one. Hence, the network possesses two modes, and the solution will contain two constants of integration. The two initial conditions are the initial currents in the two meshes. If we use the method of successive initial derivatives for the evaluation of the normal function amplitudes, then the first initial derivative of current in each mesh will have to be found. These together with the initial currents will suffice to determine all the amplitudes. Let us make a numerical example out of the above by assigning the following values to the parameters:

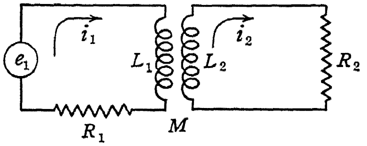

FIG. 74

$$R_1 = 5 \quad L_1 = 0.001 \quad R_2 = 10 \quad L_2 = 0.002 \quad M = 0.001.$$

Further let us suppose that:

$$e_1 = E = \text{constant}.$$

This constant voltage is suddenly inserted into the first mesh at $t = 0$.

The steady-state currents may be written down by inspection. They are:

$$\left.\begin{aligned} (i_1)_s &= \frac{E}{5} \\ \text{and:} \qquad (i_2)_s &= 0. \end{aligned}\right\} \tag{469}$$

Next we must set up the determinantal equation. This is:

$$D(p) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = 0. \tag{470}$$

For our case we have:

$$\left.\begin{aligned} c_{11} &= 0.001\,p + 5 \\ c_{12} &= 0.001\,p = c_{21} \\ c_{22} &= 0.002\,p + 10 \end{aligned}\right\} \tag{471}$$

so that:

$$D(p) = (.001\,p + 5)\,(.002\,p + 10) - (.001\,p)^2.$$

The determinantal equation is therefore:

$$p^2 + 2 \cdot 10^4\,p + 5 \cdot 10^7 = 0. \tag{472}$$

It is not necessary to multiply $D(p)$ by any power of p in forming the determinantal equation because neither mesh contains independent capacity.

The roots of (472) are the desired modes. These are:

$$\left.\begin{aligned} p_1 &= -17070 \\ p_2 &= -\ \ 2930. \end{aligned}\right\} \tag{473}$$

Hence the transient solutions are:

$$\left.\begin{aligned} (i_1)_t &= J_1^{(1)}e^{-17070\,t} + J_1^{(2)}e^{-2930\,t} \\ \text{and:} \qquad (i_2)_t &= J_2^{(1)}e^{-17070\,t} + J_2^{(2)}e^{-2930\,t}. \end{aligned}\right\} \tag{474}$$

The complete solutions are:

$$\left.\begin{aligned} i_1 &= \frac{E}{5} + J_1^{(1)}e^{-17070\,t} + J_1^{(2)}e^{-2930\,t} \\ i_2 &= 0 + J_2^{(1)}e^{-17070\,t} + J_2^{(2)}e^{-2930\,t}. \end{aligned}\right\} \tag{475}$$

For the determination of the J's by the method of successive initial derivatives, we must write these equations and their first derivatives for $t = 0$. This will give two equations for each mesh-current. Thus, we obtain for i_1:

$$\left.\begin{aligned} J_1^{(1)} + \qquad J_1^{(2)} &= i_{10} - 0.2\,E \\ -17070\,J_1^{(1)} - 2930\,J_1^{(2)} &= \left(\frac{di_1}{dt}\right)_0 \end{aligned}\right\} \tag{476}$$

and for i_2:

$$\left.\begin{aligned} J_2^{(1)} + \qquad J_2^{(2)} &= i_{20} \\ -17070\,J_2^{(1)} - 2930\,J_2^{(2)} &= \left(\frac{di_2}{dt}\right)_0 . \end{aligned}\right\} \tag{477}$$

The quantities i_{10} and i_{20} are given initial conditions. The initial derivatives $\left(\frac{di_1}{dt}\right)_0$ and $\left(\frac{di_2}{dt}\right)_0$ must be determined before the equations (476) and (477) may be solved for the normal function amplitudes. This determination proceeds from the equilibrium equations of the system, which are:

$$\left.\begin{aligned} .001\frac{di_1}{dt} + 5\,i_1 + .001\frac{di_2}{dt} &= E \\ .001\frac{di_1}{dt} + .002\frac{di_2}{dt} + 10\,i_2 &= 0. \end{aligned}\right\} \tag{478}$$

For $t = 0$, we have:

$$\left.\begin{aligned} .001\left(\frac{di_1}{dt}\right)_0 + .001\left(\frac{di_2}{dt}\right)_0 &= E - 5\,i_{10} \\ .001\left(\frac{di_1}{dt}\right)_0 + .002\left(\frac{di_2}{dt}\right)_0 &= -10\,i_{20}. \end{aligned}\right\} \tag{479}$$

These equations may be solved for the desired initial derivatives.

We might have used the recursion formula (451) to set up these equations (479) directly. Since in this example there are no elastances, (451) becomes:

$$\sum_{k=1}^{2} L_{sk}u_k^r = e_s^{r-1} - \sum_{k=1}^{2} R_{sk}u_k^{r-1}. \tag{480}$$

We need consider here only the value $r = 1$ because we want only the first initial derivatives. Writing (480) successively for $s = 1$ and $s = 2$, we get:

$$\left.\begin{aligned} L_{11}u_1^1 + L_{12}u_2^1 &= e_1^0 - R_{11}u_1^0 - R_{12}u_2^0 \\ L_{21}u_1^1 + L_{22}u_2^1 &= e_2^0 - R_{21}u_1^0 - R_{22}u_2^0. \end{aligned}\right\} \tag{481}$$

But:

$$\begin{aligned} e_1^0 &= E \\ e_2^0 &= 0 \\ R_{12} = R_{21} &= 0 \end{aligned}$$

so that we have numerically:

$$\left.\begin{aligned} .001\, u_1^1 + .001\, u_2^1 &= E - 5\, u_1^0 \\ .001\, u_1^1 + .002\, u_2^1 &= -10\, u_2^0 \end{aligned}\right\} \tag{482}$$

which is identical with the system (479).

The reader will probably very much prefer the first method for setting up these equations, because the latter method involves a more complicated form of notation. In a simple problem like the present one it would be unnecessary to use (451) except for getting the practice in applying the symbolic notation involved. Practice on simple problems such as this, will give the student the proficiency in the use of summation signs and summation indices necessary to easily understand the more advanced methods to which we shall come later.

Let us proceed with our problem. For simplicity we will assume initial rest conditions, i.e., that:

$$i_{10} = i_{20} = 0. \tag{483}$$

Then the solution of (479) or (482) gives:

and:

$$\left.\begin{aligned} \left(\frac{di_1}{dt}\right)_0 &= 2000\, E \\ \left(\frac{di_2}{dt}\right)_0 &= -1000\, E. \end{aligned}\right\} \tag{484}$$

Substituting these into (476) and (477), we have:

$$\left.\begin{aligned} J_1^{(1)} + \qquad J_1^{(2)} &= -0.2\, E \\ -17070\, J_1^{(1)} - 2930\, J_1^{(2)} &= 2000\, E \end{aligned}\right\} \tag{485}$$

and:

$$\left.\begin{aligned} J_2^{(1)} + \qquad J_2^{(2)} &= 0 \\ -17070\, J_2^{(1)} - 2930\, J_2^{(2)} &= -1000\, E. \end{aligned}\right\} \tag{486}$$

Their solutions are:

$$J_1^{(1)} = -0.1\,E \qquad J_1^{(2)} = -0.1\,E \tag{485a}$$

and:

$$J_2^{(1)} = 0.0707\,E \qquad J_2^{(2)} = -0.0707\,E. \tag{486a}$$

Hence, the complete solutions (475) are:

$$i_1 = 0.2\,E - 0.1\,E(e^{-17070t} + e^{-2930t}) \tag{487}$$

and:

$$i_2 = 0.0707\,E(e^{-17070t} - e^{-2930t}). \tag{488}$$

These results are plotted in Fig. 75 (*a*) and (*b*) respectively.

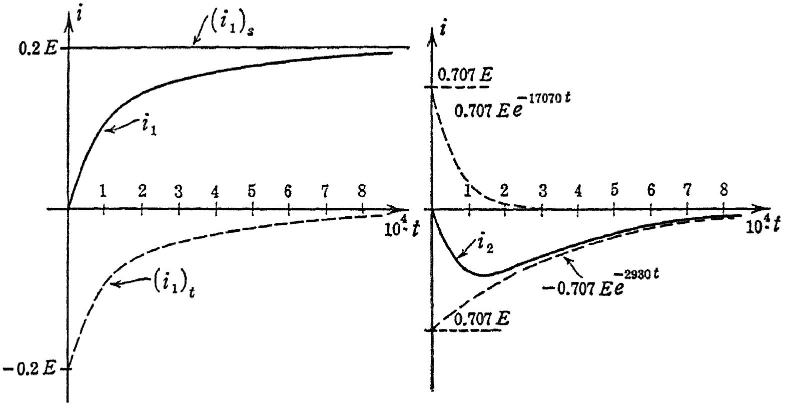

FIG. 75

For the sake of completeness let us determine the constants of integration for this problem by the direct method also. In terms of the notation for this method we have:

$$\left.\begin{aligned} i_1 &= 0.2\,E + C_{11}^{(1)}G^{(1)}e^{-17070t} + C_{11}^{(2)}G^{(2)}e^{-2930t} \\ i_2 &= 0 + C_{12}^{(1)}G^{(1)}e^{-17070t} + C_{12}^{(2)}G^{(2)}e^{-2930t} \end{aligned}\right\} \tag{489}$$

where the constants $G^{(1)}$ and $G^{(2)}$ are to be determined from the two initial conditions, and the $C_{rk}^{(\nu)}$'s are the minors of the determinant:

$$D(p) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$$

for the mode $p = p_\nu$ as indicated by the superscript. By inspection we get:

$$\left.\begin{aligned} C_{11}^{(\nu)} &= c_{22}^{(\nu)} = .002\,p_\nu + 10 \\ C_{12}^{(\nu)} &= -c_{21}^{(\nu)} = -.001\,p_\nu. \end{aligned}\right\} \tag{490}$$

Hence:

$$\left.\begin{aligned} C_{11}^{(1)} &= -24.14 \qquad & C_{11}^{(2)} &= 4.14 \\ C_{12}^{(1)} &= 17.07 \qquad & C_{12}^{(2)} &= 2.93. \end{aligned}\right\} \tag{491}$$

Assuming initial rest conditions again, we get by writing (489) for $= 0$:

$$\left.\begin{aligned} -24.14\, G^{(1)} + 4.14\, G^{(2)} &= -0.2\, E \\ 17.07\, G^{(1)} + 2.93\, G^{(2)} &= 0 \end{aligned}\right\} \tag{492}$$

which gives:

$$G^{(1)} = 4.14 \cdot 10^{-3}E \qquad G^{(2)} = -2.414 \cdot 10^{-2}E. \tag{493}$$

Substituting these values back into (489), we have the desired solutions:

$$\left.\begin{aligned} i_1 &= 0.2\, E - 0.1\, E(e^{-17070\, t} + e^{-2930\, t}) \\ i_2 &= 0 + 0.0707\, E(e^{-17070\, t} - e^{-2930\, t}) \end{aligned}\right\} \tag{494}$$

which check with the solutions (487) and (488) found by the other method.

Note that the last method used above does not require the determination of any successive initial derivatives. It evaluates the true integration constants $G^{(1)}$ and $G^{(2)}$ directly from the initial conditions $i_{10} = i_{20} = 0$. It is for this reason that it is called the direct method, as mentioned previously.

In the present case only one initial derivative had to be found for the evaluation of the normal function amplitudes by the method of successive initial derivatives. Hence there is little choice between that method and the direct method. Both are simple. However, when the network is more complicated, the direct method may have the advantage over the other.

For the sake of interest, let us calculate several more terms in the Taylor expansions for the mesh-currents in this case. We already have the first initial derivatives as given by (484). For the determination of additional initial derivatives, we have by (480):

$$\left.\begin{aligned} L_{11}u_1^{r} + L_{12}u_2^{r} &= -R_{11}u_1^{r-1} \\ L_{21}u_1^{r} + L_{22}u_2^{r} &= -R_{22}u_2^{r-1} \end{aligned}\right\} \quad (r > 1) \tag{495}$$

or:

$$\left.\begin{aligned} .001\, u_1^{r} + .001\, u_2^{r} &= -5\, u_1^{r-1} \\ .001\, u_1^{r} + .002\, u_2^{r} &= -10\, u_2^{r-1} \end{aligned}\right\} \quad (r > 1). \tag{495a}$$

These equations may be used for any value of r greater than unity. For the latter value the voltage appears on the right-hand side of the first equation as given by (482). For successive values of r the voltage term drops out because its derivative is zero. In the case of a harmonic impressed force, these derivatives are not zero, and the voltage term appears in the successive equations as well as in the first.

Writing (495a) for $r = 2$, we have:

$$\left.\begin{aligned} u_1^2 + u_2^2 &= -10^7E \\ u_1^2 + 2\,u_2^2 &= 10^7E. \end{aligned}\right\} \tag{496}$$

This gives:

$$u_1^2 = -3\cdot 10^7E \qquad u_2^2 = 2\cdot 10^7E. \tag{496a}$$

Hence (495a) for $r = 3$ becomes:

$$\left.\begin{aligned} u_1^3 + u_2^3 &= 1.5\cdot 10^{11}E \\ u_1^3 + 2\,u_2^3 &= -2\cdot 10^{11}E. \end{aligned}\right\} \tag{497}$$

This gives:

$$u_1^3 = 5\cdot 10^{11}E \qquad u_2^3 = -3.5\cdot 10^{11}E. \tag{497a}$$

For the fourth initial derivatives we have:

$$\left.\begin{aligned} u_1^4 + u_2^4 &= -2.5\cdot 10^{15}E \\ u_1^4 + 2\,u_2^4 &= 3.5\cdot 10^{15}E \end{aligned}\right\} \tag{498}$$

so that:

$$u_1^4 = -8.5\cdot 10^{15}E \qquad u_2^4 = 6\cdot 10^{15}E. \tag{498a}$$

Hence for the fifth derivatives:

$$\left.\begin{aligned} u_1^5 + u_2^5 &= 4.25\cdot 10^{19}E \\ u_1^5 + 2\,u_2^5 &= -6\cdot 10^{19}E \end{aligned}\right\} \tag{499}$$

from which:

$$u_1^5 = 1.45\cdot 10^{20}E \qquad u_2^5 = -1.025\cdot 10^{20}E. \tag{499a}$$

The Taylor expansions for the currents may therefore be written:

$$i_1 = E\left(2\cdot 10^3t - \frac{3\cdot 10^7}{2!}t^2 + \frac{5\cdot 10^{11}}{3!}t^3 - \frac{8.5\cdot 10^{15}}{4!}t^4 + \frac{1.45\cdot 10^{20}}{5!}t^5 - \cdots\right) \tag{500}$$

$$i_2 = E\left(-10^3t + \frac{2\cdot 10^7}{2!}t^2 - \frac{3.5\cdot 10^{11}}{3!}t^3 + \frac{6\cdot 10^{15}}{4!}t^4 - \frac{1.025\cdot 10^{20}}{5!}t^5 + \cdots\right) \tag{501}$$

It is quite evident that they converge very poorly. The above five terms suffice to calculate the behavior only up to about $t = 1.5 \cdot 10^{-4}$ seconds. This is still considerably less than the largest time constant involved. Hence, the Taylor expansion method of solution is very unsatisfactory in this case.

Let us turn our attention to another network for which the Taylor expansion may very well be used to represent the form of the solution during the initial stages. This is illustrated in Fig. 76. It is the same as the network given in Fig. 74 except for the introduction of the elastance in the first mesh. Let us keep the same values for the corresponding inductances and resistances as in the preceding problem, and introduce the value $S_1 = 10^8$ darafs for the elastance. We will proceed to obtain the normal function solution first.

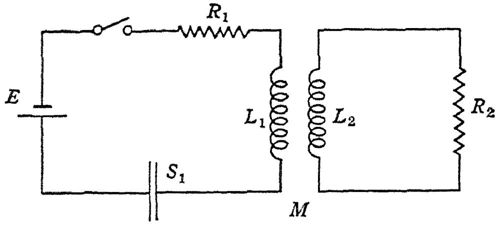

FIG. 76

The first mesh has a weight of two, and the second has a weight of one. Hence, the network possesses three modes, and the determinantal equation will be a cubic. We have:

$$\left.\begin{aligned} c_{11} &= .001\,p + 5 + 10^8 p^{-1} \\ c_{12} &= .001\,p = c_{21} \\ c_{22} &= .002\,p + 10. \end{aligned}\right\} \tag{502}$$

Hence:

$$D(p) = (.001\,p + 5 + 10^8 p^{-1})(.002\,p + 10) - (.001\,p)^2. \tag{503}$$

The first mesh contains independent capacitance, so that $D(p)$ must be multiplied through by p in forming the determinantal equation. We have:

$$p^3 + 2 \cdot 10^4 p^2 + 2 \cdot 10^{11} p + 10^{15} = 0. \tag{504}$$

We expect from the relatively small resistances that the network is fully oscillatory, i.e., that it possesses one real mode and one pair of conjugate complex modes. The first approximation to the real root of (504) may be found by neglecting all but the last two terms, thus:

$$2 \cdot 10^{11} p + 10^{15} = 0.$$

This gives:

$$p = -5000. \tag{505}$$

If we divide this root into (504), we find that the remainder is negligible. Hence (505) is already close enough to the correct value. The remaining quadratic is:

$$p^2 + 1.5 \cdot 10^4 p + 2 \cdot 10^{11} = 0. \tag{506}$$

Its roots are:

$$p = -7500 \pm j\,447000.$$

Hence the modes of our given network may be summarized as follows:

$$\left.\begin{aligned} p_1 &= -5000 \\ p_2 &= -7500 + j\,447000 \\ p_3 &= -7500 - j\,447000. \end{aligned}\right\} \tag{507}$$

Since there are no steady-state currents in either of the meshes, the complete solutions for current after closing the switch ($t = 0$) become:

$$\left.\begin{aligned} i_1 &= J_1^{(1)} e^{-5000\,t} + (J_1^{(2)} e^{j\,447000\,t} + J_1^{(3)} e^{-j\,447000\,t}) e^{-7500\,t} \\ i_2 &= J_2^{(1)} e^{-5000\,t} + (J_2^{(2)} e^{j\,447000\,t} + J_2^{(3)} e^{-j\,447000\,t}) e^{-7500\,t} \end{aligned}\right\} \tag{508}$$

or in terms of the true integration constants we have:

$$\left.\begin{aligned} i_1 &= C_{11}^{(1)} G^{(1)} e^{-5000\,t} + (C_{11}^{(2)} G^{(2)} e^{j\,447000\,t} + C_{11}^{(3)} G^{(3)} e^{-j\,447000\,t}) e^{-7500\,t} \\ i_2 &= C_{12}^{(1)} G^{(1)} e^{-5000\,t} + (C_{12}^{(2)} G^{(2)} e^{j\,447000\,t} + C_{12}^{(3)} G^{(3)} e^{-j\,447000\,t}) e^{-7500\,t} \end{aligned}\right\} \tag{509}$$

where $C_{11}^{(\nu)}$ and $C_{12}^{(\nu)}$ are the minors of:

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \tag{510}$$

for the mode $p = p_\nu$.

We have three integration constants to evaluate, namely $G^{(1)}$, $G^{(2)}$, and $G^{(3)}$. For this purpose we have three initial conditions to specify, namely i_{10}, i_{20}, and q_{10}. We will proceed with the direct method of evaluation first. The initial current conditions may be substituted into the equations (509) directly. The initial charge in mesh #1 cannot be substituted until we have formed the charge equation for mesh #1. This is done by integrating the corresponding current equation once with respect to time. This gives:

$$q_1 = \frac{C_{11}{}^{(1)}G^{(1)}}{-5000}e^{-5000t} + \left(\frac{C_{11}{}^{(2)}G^{(2)}e^{j\,447000\,t}}{-7500 + j\,447000} + \frac{C_{11}{}^{(3)}G^{(3)}e^{-j\,447000\,t}}{-7500 - j\,447000}\right)e^{-7500t} + 10^{-8}E. \qquad (511)$$

Note the introduction of the term $10^{-8}E$ in the above equation. This is the steady-state charge in mesh #1. The steady-state current is zero, but the charge is equal to the capacity times the steady voltage E. If the impressed force had been harmonic instead of constant, then the current equation would have contained a steady-state term, and the corresponding steady-state charge would have been the integral of the current term as given by the first equation (441). It is only in the case of a constant-applied voltage that the steady-state charge must be inserted by inspection, as was done above. If there is any doubt about what the form of this term is, the steady-state charge should be determined for a harmonic-impressed force having the same magnitude, and the limit: Frequency → 0 carried out. This is hardly ever necessary however, since the steady-state charge is always equal to the contour capacity times the voltage impressed on the contour of the mesh in question. If there is no steady-state impressed voltage in that mesh, then the steady-state mesh charge is zero.

We will assume initial rest conditions for the evaluation of the integration constants. Then setting $t = 0$ in equations (509) and (511), we have:

$$\left.\begin{aligned} &C_{11}{}^{(1)}G^{(1)} + C_{11}{}^{(2)}G^{(2)} + C_{11}{}^{(3)}G^{(3)} = 0 \\ &C_{12}{}^{(1)}G^{(1)} + C_{12}{}^{(2)}G^{(2)} + C_{12}{}^{(3)}G^{(3)} = 0 \\ &\frac{C_{11}{}^{(1)}G^{(1)}}{-5000} + \frac{C_{11}{}^{(2)}G^{(2)}}{-7500 + j\,447000} + \frac{C_{11}{}^{(3)}G^{(3)}}{-7500 - j\,447000} = -10^{-8}E. \end{aligned}\right\} \qquad (512)$$

Now from (510) we find:

$$\left.\begin{aligned} C_{11} &= c_{12} \\ C_{12} &= -c_{21} \end{aligned}\right\} \qquad (513)$$

so that:

$$\left.\begin{aligned} C_{11}^{(\nu)} &= .002\, p_\nu + 10 \\ C_{12}^{(\nu)} &= -.001\, p_\nu. \end{aligned}\right\} \qquad (514)$$

Substituting the values for the modes, as given by (507), into (514), we have:

$$\left.\begin{aligned} C_{11}^{(1)} &= 0 \quad & C_{11}^{(2)} &= -5 + j\,894 \quad & C_{11}^{(3)} &= -5 - j\,894 \\ C_{12}^{(1)} &= 5 \quad & C_{12}^{(2)} &= 7.5 - j\,447 \quad & C_{12}^{(3)} &= 7.5 + j\,447 \end{aligned}\right\} \qquad (515)$$

and:

$$\left.\begin{aligned} \frac{C_{11}^{(2)}}{-7500 + j\,447000} &= 2 \cdot 10^{-3} - j\,2.24 \cdot 10^{-5} \\ \frac{C_{11}^{(3)}}{-7500 - j\,447000} &= 2 \cdot 10^{-3} + j\,2.24 \cdot 10^{-5}. \end{aligned}\right\} \qquad (516)$$

Note that the minors that are functions of a pair of conjugate modes, are themselves conjugates. The equations (512) now become:

$$\left.\begin{aligned} 0 \quad &+ (-5 + j\,894)G^{(2)} + (-5 - j\,894)G^{(3)} = 0 \\ 5\,G^{(1)} &+ (7.5 - j\,447)G^{(2)} + (7.5 + j\,447)G^{(3)} = 0 \\ 0 \quad &+ (2 - j\,.0224)G^{(2)} + (2 + j\,.0224)G^{(3)} = -10^{-5}E. \end{aligned}\right\} \qquad (517)$$

The solution of this system is given by:

$$\left.\begin{aligned} G^{(1)} &= 5 \cdot 10^{-6}\,E \\ G^{(2)} &= (-2.5 \cdot 10^{-6} + j\,1.4 \cdot 10^{-8})\,E \\ G^{(3)} &= (-2.5 \cdot 10^{-6} - j\,1.4 \cdot 10^{-8})\,E. \end{aligned}\right\} \qquad (518)$$

Substituting these values back into the equations (509) and simplifying, we have:

$$\left.\begin{aligned} i_1 &= 4.47 \cdot 10^{-3}\,Ee^{-7500t} \cdot \sin 447000\,t \\ i_2 &= 2.5 \cdot 10^{-5}\,Ee^{-5000t} \\ &\qquad - 2.235 \cdot 10^{-3}\,Ee^{-7500t} \sin (447000\,t + .0112). \end{aligned}\right\} \qquad (519)$$

Note that on account of $C_{11}^{(1)} = 0$, $J_1^{(1)} = 0$, so that the normal function corresponding to the mode $p_1 = -5000$ does not appear in the first mesh. The reason for this is that the value -5000 for this mode is only approximate. If we had evaluated it more accurately, we should have found that mesh #1 contained this mode with perhaps a very small amplitude as compared to that of the sine term. However, the present result is certainly sufficiently accurate. The second mesh contains both modes, but the magnitude of the sine term is about one-hundred times that of the simple exponential. Hence in the second mesh the current is also practically a damped sinusoid. Its amplitude is just half of that in mesh #1. Note that both expressions (519) are zero for $t = 0$, as they should be. Also:

$$\int i_1\,dt + 10^{-8}E = 0$$

for $t = 0$, as it should on account of the condition:

$$q_{10} = 0.$$

This is a check on the accuracy of the evaluation. Such checks should always be made in work of this sort, because it is so easy to make numerical errors in the process of solving simultaneous equations with complex coefficients such as (517).

We shall now evaluate the transient current amplitudes by the method of successive derivatives so that the reader will get an idea of the comparative amount of work involved in the two processes. This method evaluates the amplitudes in the two current expressions (508) separately. Since each of these contains three amplitudes, we must form their first and second derivatives, and then consider the resulting equations for $t = 0$. Thus we get:

$$\left.\begin{aligned}
i_{10} &= J_1^{(1)} + J_1^{(2)} + J_1^{(3)} \\
\left(\frac{di_1}{dt}\right)_0 &= -5\cdot 10^3 J_1^{(1)} + (-7.5+j447)\cdot 10^3 J_1^{(2)} + (-7.5-j447)\cdot 10^3 J_1^{(3)} \\
\left(\frac{d^2i_1}{dt^2}\right)_0 &= 2.5\cdot 10^7 J_1^{(1)} + (-7.5+j447)^2\cdot 10^6 J_1^{(2)} + (-7.5-j447)^2\cdot 10^6 J_1^{(3)}
\end{aligned}\right\} \tag{520}$$

and:

$$\left.\begin{aligned}
i_{20} &= J_2^{(1)} + J_2^{(2)} + J_2^{(3)} \\
\left(\frac{di_2}{dt}\right)_0 &= -5\cdot 10^3 J_2^{(1)} + (-7.5+j447)\cdot 10^3 J_2^{(2)} + (-7.5-j447)\cdot 10^3 J_2^{(3)} \\
\left(\frac{d^2i_2}{dt^2}\right)_0 &= 2.5\cdot 10^7 J_2^{(1)} + (-7.5+j447)^2\cdot 10^6 J_2^{(2)} + (-7.5-j447)\cdot 10^6 J_2^{(3)}.
\end{aligned}\right\} \tag{521}$$

Before these systems can be solved for the desired amplitudes, we must determine separately the first and second initial derivatives of the two currents. For this purpose we can either consider the system of inhomogeneous differential equations expressing the dynamic equilibrium of the network, or make use of the recursion formula (451). We will use the former method here because it is more illustrative. We have for the network of Fig. 76:

$$\left.\begin{aligned} L_1 \frac{di_1}{dt} + R_1 i_1 + S_1 \int i_1\,dt + M\frac{di_2}{dt} &= E \\ M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + R_2 i_2 &= 0. \end{aligned}\right\} \qquad (522)$$

Considering these for $t = 0$, and assuming initial rest conditions as in the previous method, we have numerically:

$$\left.\begin{aligned} .001\left(\frac{di_1}{dt}\right)_0 + .001\left(\frac{di_2}{dt}\right)_0 &= E \\ .001\left(\frac{di_1}{dt}\right)_0 + .002\left(\frac{di_2}{dt}\right)_0 &= 0. \end{aligned}\right\} \qquad (523)$$

This gives:

$$\left.\begin{aligned} \left(\frac{di_1}{dt}\right)_0 &= 2000\,E \\ \text{and:}\qquad \left(\frac{di_2}{dt}\right)_0 &= -1000\,E. \end{aligned}\right\} \qquad (524)$$

Now differentiating the equations (522) once, and again setting $t = 0$, we get by using the values (524) just found:

$$\left.\begin{aligned} .001\left(\frac{d^2i_1}{dt^2}\right)_0 + .001\left(\frac{d^2i_2}{dt^2}\right)_0 &= -10^4 \\ .001\left(\frac{d^2i_1}{dt^2}\right)_0 + .002\left(\frac{d^2i_2}{dt^2}\right)_0 &= 10^4. \end{aligned}\right\} \qquad (525)$$

These give:

$$\left.\begin{aligned} \left(\frac{d^2i_1}{dt^2}\right)_0 &= -3\cdot 10^7\,E \\ \text{and}\qquad \left(\frac{d^2i_2}{dt^2}\right)_0 &= 2\cdot 10^7\,E. \end{aligned}\right\} \qquad (526)$$

Substituting the values given by (524) and (526), together with the rest conditions $i_{10} = i_{20} = 0$, into the systems (520) and (521), these are ready to be solved by the usual methods. They yield:

$$\left.\begin{aligned} J_1^{(1)} &= 0 \\ J_1^{(2)} &= -j\,2.24 \cdot 10^{-3}\,E \\ J_1^{(3)} &= +j\,2.24 \cdot 10^{-3}\,E \end{aligned}\right\} \tag{527}$$

and:

$$\left.\begin{aligned} J_2^{(1)} &= 2.5 \cdot 10^{-5}\,E \\ J_2^{(2)} &= (-1.25 \cdot 10^{-5} + j\,1.117 \cdot 10^{-3})\,E \\ J_2^{(3)} &= (-1.25 \cdot 10^{-5} - j\,1.117 \cdot 10^{-3})\,E \end{aligned}\right\} \tag{528}$$

respectively. Inserting these results in the equations (508), we get the same final forms as those obtained by the direct method and given by (519).

In this particular example the latter method consumes more time and space than the direct method. This is not always the case. There are problems for which the successive derivatives give the desired result more speedily. To offset the determination of successive initial derivatives, the direct method requires the calculation of minors of the determinant $D(p)$. The writer generally prefers the direct method because it is more straightforward and usually somewhat shorter, although there may be room for argument on this point.

We now wish to conclude the solution to this problem by calculating the terms in the Taylor expansions for the two mesh-currents. We already have two initial derivatives as given by (524) and (526). Let us determine several more. We will use the recursion formula (451) for this purpose. For the derivatives above the first this becomes in our case:

$$\left.\begin{aligned} L_{11}u_1^r + L_{12}u_2^r &= -R_{11}u_1^{r-1} - S_{11}u_1^{r-2} \\ L_{21}u_1^r + L_{22}u_2^r &= -R_{22}u_2^{r-1}. \end{aligned}\right\} \tag{529}$$

Substituting numerical values, we have:

$$\left.\begin{aligned} u_1^r + \,u_2^r &= -5 \cdot 10^3\,u_1^{r-1} - 10^{11}\,u_1^{r-2} \\ u_1^r + 2\,u_2^r &= -10^4\,u_2^{r-1}. \end{aligned}\right\} \tag{530}$$

For the third initial derivatives we put $r = 3$, thus:

$$\left.\begin{aligned} u_1^3 + u_2^3 &= -2 \cdot 10^{14}\,E \\ u_1^3 + 2\,u_2^3 &= -2 \cdot 10^{11}\,E. \end{aligned}\right\} \tag{531}$$

This gives:

$$u_1^3 = -4 \cdot 10^{14}\,E \qquad u_2^3 = 2 \cdot 10^{14}\,E. \tag{532}$$

Hence (530) for $r = 4$ becomes:

$$\left.\begin{aligned} u_1{}^4 + \quad u_2{}^4 &= \quad 5 \cdot 10^{18}\,E \\ u_1{}^4 + 2\,u_2{}^4 &= -2 \cdot 10^{18}\,E \end{aligned}\right\} \tag{533}$$

so that:

$$u_1{}^4 = 1.2 \cdot 10^{19}\,E \qquad u_2{}^4 = -7 \cdot 10^{18}\,E. \tag{534}$$

For $r = 5$ we have:

$$\left.\begin{aligned} u_1{}^5 + \quad u_2{}^5 &= 4 \cdot 10^{25}\,E \\ u_1{}^5 + 2\,u_2{}^5 &= 7 \cdot 10^{22}\,E. \end{aligned}\right\} \tag{535}$$

Hence:

$$u_1{}^5 = 8 \cdot 10^{25}\,E \qquad u_2{}^5 = -4 \cdot 10^{25}\,E. \tag{536}$$

The Taylor expansions to five terms therefore become:

$$i_1 = E\left(2 \cdot 10^3\,t - \frac{3 \cdot 10^7}{2!}t^2 - \frac{4 \cdot 10^{14}}{3!}t^3 + \frac{1.2 \cdot 10^{19}}{4!}t^4 + \frac{8 \cdot 10^{25}}{5!}t^5 - \cdots\right) \tag{537}$$

and:

$$i_2 = E\left(-10^3 t + \frac{2 \cdot 10^7}{2!}t^2 + \frac{2 \cdot 10^{14}}{3!}t^3 - \frac{7 \cdot 10^{18}}{4!}t^4 - \frac{4 \cdot 10^{25}}{5!}t^5 + \cdots\right). \tag{538}$$

Recalling the explicit solutions (519), we note that they are substantially damped sinusoids. If they were sinusoids alone and not exponentially damped, then the Taylor expansions above would be restricted to the odd terms only. The even terms in (537) and (538) are due to the exponential factors in the solutions (519). Since these factors are relatively unimportant so far as the general character of the oscillations during the first few cycles is concerned, we expect that the even terms in the expansions (537) and (538) are also relatively unimportant. This is the case, as we may verify by plotting the terms separately, as shown in Fig. 77 for i_1. Here the curves for the separate terms are numbered accordingly. The heavy curve is the sum of the individual curves and represents the current i_1 for the five terms in (537). We see that the first half-cycle of the oscillation is fairly well represented. The series (538) for i_2 could be plotted in the same way. If we had merely been interested in knowing what the general character of the behavior was, and how large the amplitude of the initial oscillation would be, then the above expansions could have given the desired information. They also indicate approximately what the frequency of the oscillation is, since the period can be estimated from the figure. In such a case it would be unnecessary to obtain a complete solution in normal function form. If the network contained more meshes than two, the saving in time and labor would be quite considerable, since the solution in series form eliminates the necessity of determining the modes as well as of evaluating the constants of integration. It is well, therefore, to keep this method of analysis in mind when making preliminary or approximate network investigations.

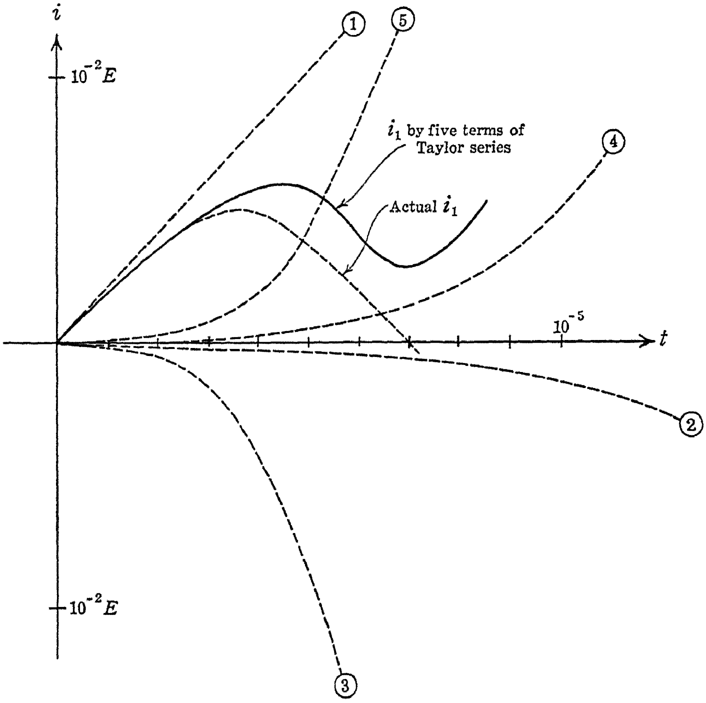

FIG. 77

7. Summary. There are other methods which may be used to evaluate the transient current amplitudes in the complete network solutions. One outstanding method is expressed by Heaviside's famous expansion formula to which we shall come in a later chapter. The others are chiefly in the form of modifications of the above fundamental methods. Each individual will develop short cuts for himself which serve to cut down the tedious manipulation involved. In connection with the method of evaluation by successive initial derivatives, a number of the latter may frequently be found by inspection of the network directly. The first initial derivatives can very often be determined that way. For example, in the case of the network of Fig. 74 we recognize that the resistances do not count so far as the initial derivatives are concerned. Also, the mutual inductance may be replaced by a common self-inductance for this purpose. We then have the circuit shown in Fig. 78 which consists essentially of two inductances of one milli-henry each in parallel. Viewed from the first mesh, this is like one inductance having a value of one-half a milli-henry. The initial derivative in the first mesh must therefore be:

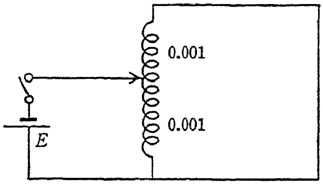

FIG. 78

$$\left(\frac{di_1}{dt}\right)_0 = \frac{E}{.0005} = 2000\,E$$

which agrees with the result (484). The initial current splits evenly between the two coils which makes the initial derivative in mesh #2 half of what it is in the first mesh. The positive clockwise assumption for the currents makes the initial derivative in the second mesh negative.

The writer does not attach much significance to these methods, and particularly does not recommend them to the beginner since they are very apt to lead him into errors, especially in the matter of algebraic sign. Proficiency in such short cuts as these must come after more experience with networks in general has been had. In the end they do not save any appreciable amount of time anyway. The method outlined above for determining successive initial derivatives is very rapid and also accurate. It can be applied by the beginner without much chance for error. In the case of higher derivatives on the other hand, the inspection method fails altogether, and the rigorous method used in this chapter must be resorted to.

A few words might be added regarding the use of the term *initial* *conditions*. Some writers upon this subject sometimes refer to the successive initial derivatives as initial conditions. This is inaccurate. The initial conditions are those conditions which exist in the network initially, i.e., just before the driving force engages, not just after. The successive initial derivatives are a consequence of the engagement of the force, whereas the currents and charges present in the various meshes before this force engages, are due to some previous disturbance. In short, the initial conditions are specified by that condition in which the driving force finds the network at the instant it takes hold.

Finally, it should be pointed out that the accuracy with which the integration constants may be determined differs somewhat between the two methods of evaluation in some cases. This occurs when the numerical work involves differences of very large numbers which are nearly of the same magnitude. In such cases one of the methods may be preferable as compared to the other, although no general rules can be given regarding this matter. Cases for which this situation arises are usually inherently peculiar. The direct method of evaluation is often able to throw additional light on these matters when given a geometrical interpretation. This interpretation of the homogeneous solution is also very helpful in the general case, and we wish, therefore, to take it up now in some detail.

PROBLEMS TO CHAPTER VI

6-1. An R, L, C circuit has the following values for its parameters:

$$R = 2002$$
$$L = 1.0$$
$$C = 10^{-6}.$$

This leads to a case having almost coincident modes. Evaluate the integration constants for a suddenly applied constant force, first, treating the modes as different, and second, treating the modes as though they were coincident. Plot the results in both cases and compare. This problem illustrates the fact that when modes are almost coincident, it is better from the standpoint of numerical accuracy, to treat them as coincident.

6-2. Treat the R, C circuit of Chapter II, and evaluate the integration constant by the direct method.

6-3. Treat the R, C circuit of Chapter II using the Taylor expansion method.

6-4. Treat the R, L, C circuit with constant excitation, using the direct method for evaluating the integration constants.

6–5. Treat the R, C circuit with harmonic excitation, using the direct method for evaluating the integration constant.

6–6. Treat the R, L, C circuit with harmonic excitation, using the direct method of evaluation of the integration constants, and check the result with that obtained in Chapter III. Also check the special cases considered in that chapter.

6–7. A certain three-mesh network has the following set of parameters:

$$\begin{array}{lll} L_{11} = 0.001 & L_{22} = 0.002 & L_{33} = 0.002 \\ R_{11} = 500 & R_{22} = 10 & R_{33} = 1000 \\ L_{12} = 10^{-5} & S_{22} = 10^{9} & L_{23} = 10^{-5}. \end{array}$$

A constant force is suddenly inserted into the first mesh. Determine the complete solution using the direct method of evaluation.

6–8. Determine the integration constants in Prob. **6–7** by the method of successive initial derivatives, and check with the direct method.

CHAPTER VII

THE VECTOR INTERPRETATION OF THE TRANSIENT SOLUTION

1. General Remarks. The present chapter proposes to give a geometrical significance to the solution of the homogeneous system of equilibrium conditions, i.e., to the transient or force-free solution of the network. The last two chapters have already shed some light upon this portion of the complete solution, but there are several outstanding points which still remain in a considerably cloudy condition. These points bear chiefly upon the algebraic principles involved in the homogeneous solution and their interpretation in terms of the physical network. For example, we found that non-trivial solutions to the homogeneous system of equations exist only when the determinant of that system vanishes. We could appreciate this in a general way from the form which the solution took when Cramer's rule was applied to the system (359) of Chapter V; i.e., we could see algebraically that since the numerators in these expressions vanished, the denominators would have to vanish also if there was to be any chance for a solution other than zero. But certainly this sort of reasoning is very superficial when it comes to shedding some real light upon the meaning of this procedure. Just what does the vanishing of the determinant mean anyway? What physical significance can we attach to the non-trivial solutions that arise when the determinant is caused to vanish? Why is it that the homogeneous solutions are not explicitly determined, but appear merely as ratios? What bearing might all this have upon the normal amplitudes and the normal coördinates in the mechanical sense? All these ideas are intimately related in the general significance of the natural behavior of an oscillatory system, whether it be mechanical or electrical in its physical make-up. They are ideas which must be clarified before a full appreciation of the transient behavior can be had. Unless they are clarified, the manipulations outlined in the previous chapters remain a mere set of mechanical motions without any vital significance, and are consequently a drudgery to remember. The considerations which we are about to take up are partly artificial and partly based upon the physical picture lent by the mechanical analog. Their consequence not only lends pictorial significance to the algebraic manipulations involved, but also to the actual physical phenomena which take place.

2. Vector Groups and Their Bearing upon the Homogeneous Equilibrium Conditions. We shall first take up some fundamental vector principles. Consider the homogeneous system of condition equations (359) of Chapter V. We repeat them here for convenience:

$$\left.\begin{array}{l} c_{11}J_1 + c_{12}J_2 + \cdots + c_{1n}J_n = 0 \\ c_{21}J_1 + c_{22}J_2 + \cdots + c_{2n}J_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_{n1}J_1 + c_{n2}J_2 + \cdots + c_{nn}J_n = 0. \end{array}\right\} \tag{539}$$

The J's are the normal function or transient mesh-current amplitudes, and the c's are the polynomial functions of the modes, given by:

$$c_{ik} = L_{ik}p + R_{ik} + S_{ik}p^{-1}. \tag{540}$$

We recall that these conditions (539) had to be satisfied in order that non-trivial solutions of the form (354) could exist, i.e., that the homogeneous equilibrium conditions (353) could lead to valid and useful solutions.

In terms of the modes, this represents an algebraic system of homogeneous linear simultaneous equations with the J's as unknowns. The determinant of this system, namely:

$$D(p) = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix} \tag{541}$$

may be called the **modular determinant** or the **determinant of modes.** It completely characterizes the form of the transient solution. It is, therefore, logical that we should be able to determine the general structure of the transient solution from a more thorough study of this determinant. This we do by giving it the following geometrical significance.[1]

Consider the determinant (541) by rows. Each row consists of n elements. It may be helpful to devise some means whereby these n elements are combined into a single quantity. This is done very easily if we consider these elements as components of a vector in n-dimensional space. Then each row of the determinant represents one vector in this space, while the whole determinant represents a group of n such vectors. Let us formulate this a little more definitely.

In our n-dimensional space — which is a fiction, of course — we suppose that we have a fixed Cartesian coördinate system, i.e., an orthogonal system of reference coördinates. The student is undoubtedly familiar with such reference systems in two- and three-dimensional cases. There we speak of them as the x and y axes, or the x, y, and z axes. Our fictitious space of n-dimensions simply carries this idea out for more than three dimensions. It can no longer be visualized in the same sense as the two- and three-dimensional cases, but it has nevertheless the same significance. The axes in this n-dimensional Cartesian system are not denoted by letters such as x, y . . . etc., but by numbers; i.e., we have axis 1, axis 2, etc., down to axis n. They are all at right angles to each other by definition.

Having thus defined our reference system, we specify the vectors as follows. Consider the first row of elements in (541), i.e.:

$$c_{11} \quad c_{12} \quad c_{13} \ldots c_{1n}. \tag{542}$$

These are to be looked upon as the n components of a vector c_1 such that c_{11} is the projection of c_1 upon axis 1; c_{12} is the projection of c_1 upon axis 2; etc., down to c_{1n} which is the projection of the vector c_1 upon the axis n. In general the vector c_1 makes some odd angle with respect to the reference axes so that it has components or projections upon them all.

Similarly for the second row in (541) we define the vector c_2 with the components:

$$c_{21} \quad c_{22} \quad c_{23} \ldots c_{2n}. \tag{543}$$

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[1] For a thorough discussion see Riemann Weber's Differentialgleichungen der Physik, Vol. I, pp. 39 e.s. (Pub. by F. Vieweg & Sohn, Braunschweig, 1925.)

The same applies to each row in (541) down to the last, for which we define a vector c_n with the components:

$$c_{n1} \quad c_{n2} \quad c_{n3} \ . \ . \ . \ c_{nn}. \tag{544}$$

Thus the determinant $D(p)$ is completely specified by the n vectors:

$$c_1 \quad c_2 \quad c_3 \ . \ . \ . \ c_n. \tag{545}$$

These vectors are said to form a **group,** so that the determinant (541) is specified by this **vector group.** By means of this process we have already accomplished something worth while, namely the modular determinant is now characterized by n things instead of n^2 things. The vector group which characterizes $D(p)$ also completely specifies the form of the transient solution.

It should be recognized that the number of dimensions or components per vector also equals the number of vectors in the group. This number is always equal to the number of meshes in the network, or to the number of degrees of freedom of the system.

3. Linear Dependence and Independence. Before proceeding with the application of these new ideas, we must present some fundamental group principles which we wish to make use of. Let us consider a two-dimensional situation to start with so that the process of visualization becomes easier. In Fig. 79 we show a two-dimensional Cartesian system with axes 1 and 2, and three arbitrary vectors a, b, and c with components a_1, a_2; b_1, b_2; and c_1, c_2 respectively. Suppose that any two of these vectors, say a and b, are given. Then it is possible to express the third vector c in terms of these two in the following form:

$$c = \alpha a + \beta b \tag{546}$$

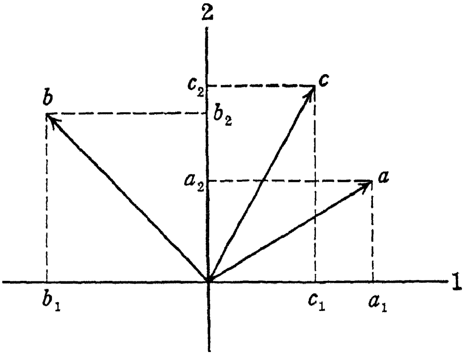


FIG. 79

where α and β are constants properly chosen to make (546) correct. That is, the vector c can be expressed as the vector sum of a certain portion of a plus a certain portion of b. It is not hard to see this geometrically. The vector c is said to be a **linear combination** of a and b, since the expression (546) is a linear one. In the same way it would be possible to express b linearly in terms of a and c, or a linearly in terms of b and c. This group of three vectors is, therefore, said to be **linearly dependent.**

Suppose now that we add a third coördinate axis (axis number 3) to the Fig. 79, perpendicular to the plane of the paper. If we lift the tip of the vector c out of the paper, then c will take on a third component on this new axis. The vectors a and b are to remain in the 1-2 plane. Now it is evidently impossible to express the vector c in terms of a and b, because c is out of the plane in which a and b lie. Similarly it would now be impossible to express a in terms of b and c, or b in terms of a and c. The group of vectors a, b, and c is now said to be **linearly independent.**

If we added a fourth arbitrary vector, say d, to this group, we would form a new group a, b, c, d. In this new group it would again be possible to express any one of the vectors linearly in terms of the other three. This group would, therefore, be linearly dependent again. There is an exception to this, namely if the vector d is so oriented as to lie in the plane specified by ab, ac, or bc, then the vectors a, b, d; a, c, d; or b, c, d respectively become dependent, and the remaining vector c, b, or a, respectively, will still be independent. We shall not consider these special cases here, however. In general if the number of vectors in the group exceeds the number of dimensions or coördinate axes, then the group is linearly dependent, like the group a, b, c in the plane of the Fig. 79. If in this figure we consider only the group formed by a and b, then this group is obviously independent because there is no way in which b can be linearly expressed in terms of a or vice versa.

In our network case we always have as many vectors as dimensions or coördinates. Hence, it is possible for these vectors to be linearly independent. But note that they do not necessarily have to be independent. Consider the two vectors a and b in the two-dimensional space of Fig. 80. They obviously form a linearly dependent group. We might, for example, have a three-dimensional case with three vectors so oriented that they all lie in a plane — any plane. They will then form a linearly dependent group. They may even all three lie along the same straight line.

Then they certainly become linearly dependent. We will correlate these ideas in the following way.

Take Fig. 80, for example. Here the number of available dimensions is two. The number of dimensions actually occupied or consumed by the vectors a and b, however, is only one. Hence, the vector group a, b occupies one less than the available number of dimensions. This is what makes it a linearly dependent group. If it had occupied all of the available dimensions, i.e., if a and b of Fig. 80 had not coincided, then their group would have been linearly independent.

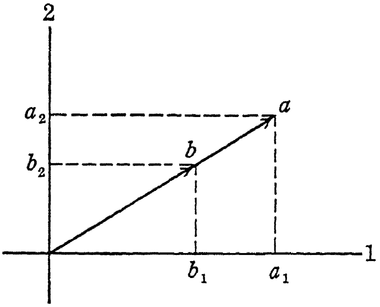


FIG. 80

Similarly if, in the case of three available dimensions (three-mesh problem), the three vectors which specify the group lie in a plane, i.e., occupy only two out of the three available dimensions, then they will be linearly dependent. If they occupy all three dimensions, i.e., do not lie in a plane, then they become linearly independent.

In general for an n-mesh or n-dimensional case involving a group of n vectors, the latter will be linearly independent if they occupy all of the available dimensions. If they occupy only $n - 1$ or less dimensions, then they become linearly dependent. Thus, for a linearly dependent group, there will be one or more dimensions still available or unoccupied.

Verbally, this situation is conveniently expressed by what is known as the **rank** of a vector group. Briefly the rank of a vector group is equal to that number of dimensions which are actually occupied by the group. For example, if in an n-dimensional problem, n dimensions are occupied by the vector group, then the latter has the rank n; if the group occupies only $n - 1$ dimensions, then its rank is $n - 1$, etc. Thus a group of rank n is linearly independent, whereas one of rank $n - 1$ or less is linearly dependent.

4. Rank of the Modular Determinant. The above ideas regarding vector groups are applied directly to the modular determinant (541). In particular the rank of the vector group is also the rank of the determinant which it represents. It remains for us to show what significance the rank of a determinant has. In order to do this we must make use of a well-known theorem regarding determinants.

Suppose we consider the determinant:

$$D = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}. \tag{547}$$

If any row in this determinant is a linear combination of the other two, then the determinant vanishes. For example, if:

$$\left.\begin{aligned} c_{11} &= \alpha c_{21} + \beta c_{31} \\ c_{12} &= \alpha c_{22} + \beta c_{32} \\ c_{13} &= \alpha c_{23} + \beta c_{33} \end{aligned}\right\} \tag{547a}$$

i.e., if the first row is a linear combination of the second and the third, then:

$$D = \begin{vmatrix} \alpha c_{21} & \alpha c_{22} & \alpha c_{23} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} + \begin{vmatrix} \beta c_{31} & \beta c_{32} & \beta c_{33} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = 0.$$

But if any row is such a linear combination of the others, then the corresponding vector group occupies at least one less than the available number of dimensions. Hence, the rank of the vector group and that of the determinant must be $n - 1$ or less.

In general we can see from the above that if the rank of a determinant is $n - 1$ or less, then that determinant vanishes. Conversely, if a determinant vanishes, its rank is $n - 1$ or less. It is also easy to see that if the determinant does not vanish, then the corresponding vector group must be linearly independent. Its rank or that of the determinant is then n.

It can further be shown that if the determinant vanishes, but at least one of its first minors does not, then the rank is $n - 1$. If the determinant and all of its first minors vanish, but at least one of its second minors does not, then the rank is $n - 2$, etc.

The converse of each of these statements is also true. Thus, the ideas regarding linear dependence and independence in the vector group, and its rank are correlated with the vanishing of the corresponding determinant and its minors.

5. Geometrical Significance of the Homogeneous Solution. Before we are able to take up the subject of this section, it will be necessary to briefly review the meaning of a **scalar product** of two vectors. Consider, for example, the vectors a and c in Fig. 79. Let us denote the angle that a makes with the horizontal by ϕ, and that for c by ψ. Then we would have:

$$\left.\begin{aligned} a_1 &= |a| \cos \phi \\ a_2 &= |a| \sin \phi \\ c_1 &= |c| \cos \psi \\ c_2 &= |c| \sin \psi. \end{aligned}\right\} \tag{548}$$

The scalar product between the vectors a and c is defined as:

$$a \cdot c = a_1c_1 + a_2c_2. \tag{549}$$

But by (548) this is:

$$\begin{aligned} a \cdot c &= |a|\,|c|\,(\cos \psi \cos \phi + \sin \psi \sin \phi) \\ &= |a|\,|c| \cos (\psi - \phi). \end{aligned} \tag{549a}$$

It is clear that $(\psi - \phi)$ is the angle between the vectors a and c. Hence, the scalar product is also expressible as the product of the magnitudes of the vectors times the cosine of the angle between them. The expression (549) for the scalar product is more general, however. The important point which (549a) illustrates is that the scalar product vanishes when the vectors are perpendicular to each other. Conversely, if the scalar product between two vectors vanishes, then they are normal to each other, or orthogonal.

This situation may easily be visualized in two- or three-dimensional space. When we speak of n dimensions, the visualization becomes more or less fictitious. Nevertheless the mathematical definition of a scalar product is still possible. Namely:

$$a \cdot c = a_1c_1 + a_2c_2 + a_3c_3 + \cdots + a_nc_n \tag{550}$$

is designated as the scalar product of the two n-dimensional vectors a and c. Furthermore if:

$$a_1c_1 + a_2c_2 + \cdots + a_nc_n = 0$$

then the two vectors are said to be orthogonal, just as in the two-dimensional case above.

Let us now consider the homogeneous system of equations (539). If we look upon the amplitudes:

$$J_1,\ J_2,\ J_3,\cdot\cdot\cdot J_n \tag{551}$$

as components of a vector J, then the first equation in the set is obviously the scalar product between the vector c_1 of the group, and J. Similarly, the second equation (539) is the scalar product between the vector c_2 of the group and J; and so forth down to the last equation which represents the scalar product between the vector c_n and J. The system of homogeneous equations (539) therefore represent the scalar products of the vector J with each vector in the group, thus:

$$\left.\begin{aligned} c_1\cdot J &= 0\\ c_2\cdot J &= 0\\ \cdot\ \cdot\ &\cdot\ \cdot\\ c_n\cdot J &= 0. \end{aligned}\right\} \tag{552}$$

This is a very much more compact way of writing the system of condition equations. It is not only compact, but is also more easily interpreted. The system (552) states in effect that the vector J, with components given by (551) must be so determined that its scalar product with each vector in the modular group shall vanish. But this means that J shall be *simultaneously orthogonal* to the vectors in the modular group. Hence, it is clear from this geometrical interpretation that the homogeneous system of equations (539) merely determines the position or direction of the vector J. *It does not specify its length.* Since the direction of J is determined by the ratios of its components (551) to one another, it is clear that these alone can be determined by the condition equations (539). This result, therefore, gives a pictorial meaning to the solution of a homogeneous system of equations, which is more satisfying than the algebraic significance pointed out in the last chapter.

Let us illustrate this with a simple example. Suppose we consider the two-dimensional numerical example:

$$\left.\begin{aligned} 5\,J_1 + 2\,J_2 &= 0\\ 3\,J_1 + 7\,J_2 &= 0. \end{aligned}\right\} \tag{553}$$

Here we have:

$$\left.\begin{aligned} c_{11} &= 5 \qquad & c_{12} &= 2 \\ c_{21} &= 3 \qquad & c_{22} &= 7. \end{aligned}\right\} \tag{554}$$

Fig. 81 illustrates the modular group of vectors c_1 and c_2. The number of available dimensions is *two*, and the number of occupied dimensions is *two*. Hence, the rank of the determinant:

$$D = \begin{vmatrix} 5 & 2 \\ 3 & 7 \end{vmatrix} \tag{555}$$

is *two*. Consequently it should not vanish, which is correct since its value is:

$$D = 5 \cdot 7 - 3 \cdot 2 = 29. \tag{556}$$

The solution to the system (553) demands that a direction in the plane of the paper of Fig. 81 be found which will be normal to both the vectors c_1 and c_2. This is manifestly impossible. Note that we cannot use the dimension perpendicular to the plane of the paper because this does not exist so far as the set (553) is concerned, because we are dealing with a two-dimensional problem. But this result checks with our algebraic conditions in regard to the existence of solutions to homogeneous systems. These, it will be recalled, state that a solution exists only when the determinant of the system vanishes. In terms of the vector group this means that solutions exist only when the rank of this group is $n - 1$ or less, i.e., when the vector group occupies at least one less than the available number of dimensions.

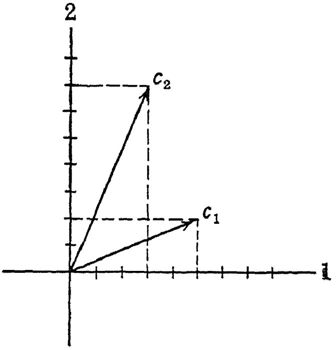


FIG. 81

Let us modify the system (553) so that this will be the case. For example, suppose we have:

$$\left.\begin{aligned} 5\,J_1 + 2\,J_2 &= 0 \\ 7\,J_1 + 2.8\,J_2 &= 0. \end{aligned}\right\} \tag{557}$$

The vector group for this system is shown in Fig. 82. We see that the vectors c_1 and c_2 lie along the same line, and hence occupy only one out of the available two dimensions. This leaves a normal direction to both vectors, which is the position to be occupied by J so that it will be the desired solution. Since the rank of the vector group is now *one*, the determinant of (557) should vanish. This it does, as may easily be seen.

Suppose now we consider the system of three equations with three unknowns:

$$\left.\begin{aligned} c_{11}J_1 + c_{12}J_2 + c_{13}J_3 &= 0 \\ c_{21}J_1 + c_{22}J_2 + c_{23}J_3 &= 0 \\ c_{31}J_1 + c_{32}J_2 + c_{33}J_3 &= 0. \end{aligned}\right\} \tag{558}$$

The modular group now consists of three vectors. If they occupy all three of the available dimensions, then it will be impossible to find a direction for J such that it will be simultaneously orthogonal to all three vectors c_1, c_2, and c_3. It is obvious that these three vectors must lie in a plane in order for a solution to be possible. Then the desired direction is found by erecting a perpendicular to this plane. But if the three vectors c_1, c_2, and c_3 lie in a plane, then they occupy only two dimensions. The rank of the group is then $n - 1$ or 2, and hence the determinant of the system of coefficients in (558) will vanish, so that the algebraic considerations are again satisfied.

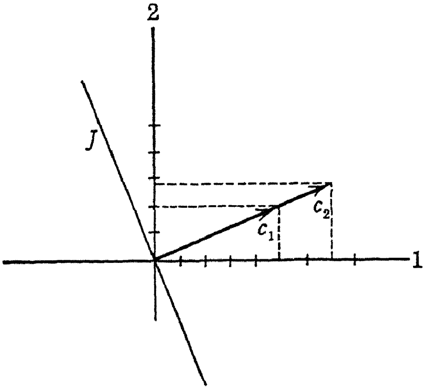


FIG. 82

Now suppose that the vectors in the group specified by (558) all lie along the same straight line. Then they will occupy only one dimension out of the available three. If this is so, then we can erect a plane perpendicular to this straight line, and having done so, any direction in this plane will be simultaneously orthogonal to the modular group. It seems as though we would then have an infinite number of solutions to our given system (558). This is true. But only two J-vectors in this plane will be linearly independent according to what was said in a previous paragraph.

Hence we have in this case two linearly independent solutions. The rank of the modular group, or of the determinant, will now be *one* or $n - 2$.

We can easily see how this reasoning will be continued for more than three available dimensions, although it can no longer be visualized as readily. If the available dimensions are n in number, and $n - k$ of them are occupied by the modular group of n vectors $c_1, c_2, c_3 \ldots c_n$, then the rank of the group is $n - k$, and there exist k linearly independent solutions.

From what has been said regarding the rank of the modular determinant, it follows that the condition for the existence of one solution is that the determinant shall vanish, but at least one of its first minors shall not. The condition for the existence of two linearly independent solutions is that the determinant and all of its first minors shall vanish, but at least one of its second minors shall not. It is clear how the conditions are extended to admit the existence of more than two linearly independent solutions.

In the case of electrical networks, more than one independent solution to the homogeneous system (539) would mean that the transient portion of the complete solution were not unique, i.e., that the natural behavior could assume two or more altogether different aspects. The condition for the existence of a unique transient behavior, therefore, is that at least one of the principal minors of the modular determinant shall not vanish. We do not have to worry about this point, however, in any practical case. Nature has seen to it that a network always behaves in the same way under identical conditions. In actual cases, therefore, it will never occur that all of the principal minors of a modular determinant for a passive network vanish. It is interesting to keep this condition for the existence of a unique solution in mind, however, in network analysis. It sometimes serves as a check on numerical work.

6. Physical Significance of the Homogeneous Solution. We have just seen that the solution to a given homogeneous system of equations specified by a modular group of vectors, defines a direction in the n-dimensional space in which this group is situated. This is, of course, speaking in terms of geometry. The question now arises: What can we gather from this so far as the physical network is concerned? In order to answer this, we must first recall that the vector group designated by the coefficients c_{ik} is fixed only in terms of the modes of the system. That is, each mode determines a new vector group. We already pointed out in Chapter V that a system of homogeneous equations such as (539) is written for each mode, and that therefore each mode introduces a corresponding normal function term in the transient solution for each mesh. Now we see that each mode specifies a corresponding coördinate or direction in our fictitious n-dimensional space. This is done in the following manner.

When we first arrive at the set of simultaneous equations (539), they have the significance of condition equations which must be satisfied in order that the assumed transient solutions may exist. The coefficients of this system, and hence the vector group thus specified, are functions of p. From the above, the vector group must be linearly dependent in order to allow the existence of a non-trivial solution. This requires the vanishing of the determinant. But the latter vanishes in general for $2n$ values of p, which are the modes. Hence for these values of p the vector group becomes linearly dependent. Each mode specifies a complete new vector group. When the modes come in pairs of conjugates, the corresponding vector groups come in such pairs. Each pair defines one decrement, one frequency, and also one direction, in the n-dimensional space, which is simultaneously orthogonal to all the vectors in the group. For the single modes, which are mere decrements, each mode specifies a linearly dependent vector group, and hence a normal direction. These normal directions thus specified for each mode or pair of modes, are called the **normal coördinates** of the system for the corresponding modes.

We spoke in an earlier chapter of these normal coördinates. We said there that to each normal coördinate there was assigned one mode or pair of modes depending upon whether the mode was a single one or in the form of a pair of conjugates. We also said that these normal coördinates were such directions in which the respective oscillations and rates of decay were isolated, i.e., singled out. Now we see that the homogeneous condition equations (539) actually determine these normal coördinates or directions.

In simple mechanical systems, these normal coördinates actually exist. Consider the mechanical system illustrated by Fig. 83, for example. Here we have a ball resting upon a plane surface (plane of the paper), held by four springs. The problem is a two-dimensional one. Hence, we describe the position of the ball at any instant by the two coördinates x_1 and x_2. If we pulled the ball out of its position of static equilibrium and let it go, it would oscillate, not back and forth in a straight line, but so as to describe some kind of peculiar motion on the 1-2 plane. The projection of the ball's motion upon the axis 1 is the force-free solution x_1 as a function of time. The projection upon the axis 2 is the force-free behavior in x_2 versus time. Both projections will in general be composed of two frequencies, with component amplitudes and phases, superposed. The same two frequencies and decrements, however, will appear in the motions upon the two axes. These frequencies and decrements are the modes of the system. Let us call them p_1 and p_2.

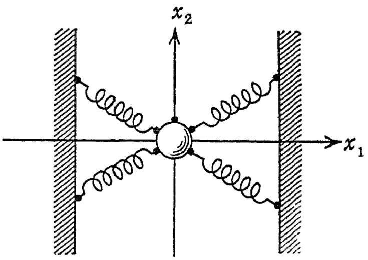


FIG. 83

If now we were to project the motion of the ball upon a movable ruler, and oriented this ruler successively in all possible directions in the plane of motion, we would eventually find two separate directions or positions of the ruler such that in one of these the motion would consist only of the one frequency and decrement defined by p_1, and in the other the motion would consist only of the one frequency and decrement defined by p_2. These two directions or coördinates so found, would be the normal coördinates corresponding to the normal modes p_1 and p_2 respectively. Their directions in the 1-2 plane would in general be arbitrary and not normal to each other. Hence, they would both have projections upon both axes 1 and 2. This is why the projected motion of the ball upon these axes consists of a superposition of both normal modes of oscillation and decay.

The analogous electrical network to the above mechanical system was discussed in Chapter I and is given in Fig. 3. It contains two meshes corresponding to the two axes in Fig. 83. For the network it is not possible to designate physical normal meshes corresponding to the normal coördinates in the mechanical system.

The analogy fails in this particular. Nevertheless we may speak of the normal coördinates or meshes in an electrical network as fictitious things. The analogy between the mechanical and electrical systems is complete regarding this matter from the mathematical standpoint, and it is, therefore, helpful to bring these fictitious things into the process of network analysis.

The concept of normal coördinates certainly clarifies the significance of the homogeneous solution. The homogeneous system of equations merely locates the normal directions, but does not fix the amplitudes of the normal modes of oscillation and decay in these directions. These amplitudes are determined from the initial conditions, of which there are just a sufficient number, as has already been pointed out.

As a matter of fact, the amplitudes of oscillation in the normal coördinates are proportional to the true integration constants which we have been denoting by $G^{(\nu)}$. The minors $C_{ik}^{(\nu)}$ are in the same proportion to the cosines of the angles between the normal coördinates and the meshes $k = 1, 2 \dots n$. That is why:

$$\left.\begin{aligned} C_{i1}^{(\nu)}G^{(\nu)} &= J_1^{(\nu)} \\ C_{i2}^{(\nu)}G^{(\nu)} &= J_2^{(\nu)} \\ \cdots & \cdots \\ C_{in}^{(\nu)}G^{(\nu)} &= J_n^{(\nu)} \end{aligned}\right\}$$

are the component amplitudes (projections) in the various meshes corresponding to the mode p_ν.

As an illustration, consider the two-mesh network of Fig. 74 in Chapter VI. Here we had:

$$\left.\begin{aligned} c_{11} &= .001\,p + 5 \\ c_{12} &= .001\,p = c_{21} \\ c_{22} &= .002\,p + 10 \end{aligned}\right\}$$

and:

$$\left.\begin{aligned} p_1 &= -17070 \\ p_2 &= -\ \ 2930. \end{aligned}\right\}$$

Hence:

$$\left.\begin{aligned} c_{11}^{(1)} &= -12.07 \\ c_{12}^{(1)} &= -17.07 \\ c_{22}^{(1)} &= -24.14 \end{aligned}\right\} \tag{559}$$

and:

$$\left.\begin{aligned} c_{11}^{(2)} &= \ \ 2.07 \\ c_{12}^{(2)} &= -2.93 \\ c_{22}^{(2)} &= \ \ 4.14. \end{aligned}\right\} \tag{560}$$

Consider the mode p_1 first. The components for the vector group corresponding to this mode are given by (559). This vector group is shown in Fig. 84. Both vectors $c_1^{(1)}$ and $c_2^{(1)}$ are coincident as they should be. The direction of the normal coördinate is at right angles to this line as indicated in the figure. It makes an angle of 35.05° with the horizontal. Suppose we denote the amplitude in this normal coördinate by N_1, as indicated in the figure. Then it must follow that:

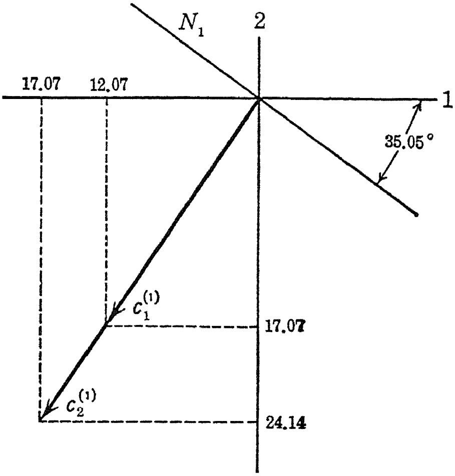


FIG. 84

$$J_1^{(1)} = N_1 \cos 35^\circ \quad \text{and:} \quad J_2^{(1)} = N_1 \sin 35^\circ \qquad (561)$$

because $J_1^{(1)}$ and $J_2^{(1)}$ are the projections of the normal amplitude upon the axes 1 and 2 respectively. But in the last chapter we found that:

$$J_1^{(1)} = 0.1\, E.$$

Hence it follows that:

$$N_1 = \frac{0.1\, E}{\cos 35^\circ} = 0.122\, E \qquad (562)$$

which is the amplitude of the mode p_1 in its normal coördinate. Its position is in the second quadrant. Its projection upon axis 2 is:

$$J_2^{(1)} = 0.122\, E \sin 35^\circ = 0.0707\, E$$

which checks with what was found previously. From this it is clear that the amplitude corresponding to the normal mode p_1 in its own coördinate is N_1, and that the projections of this amplitude upon the axes 1 and 2 give the transient mesh-current amplitudes in the meshes 1 and 2 respectively corresponding to this mode.

Considering the mode p_2 for this problem, we have the vector group whose components are specified by (560). This group is composed of $c_1^{(2)}$ with components:

$$c_{11}^{(2)} = 2.07 \qquad c_{12}^{(2)} = -2.93$$

and $c_2^{(2)}$ with components:

$$c_{21}^{(2)} = -2.93 \qquad c_{22}^{(2)} = 4.14.$$

The group is plotted in Fig. 85. The two vectors are again in line. The position of the normal coördinate is normal to this line as

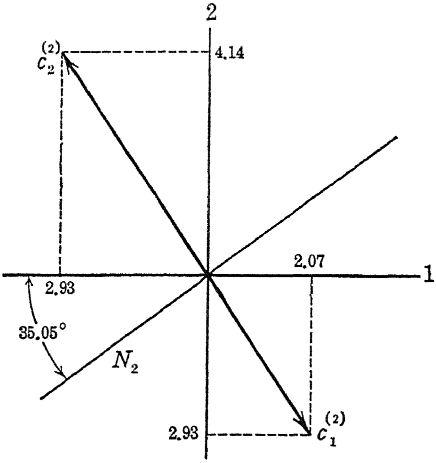


FIG. 85

shown. The projection of the amplitude N_2, in this normal coördinate, upon axis 1 is according to the last chapter:

$$J_1^{(2)} = -0.1\,E$$

so that the position of N_2 must be in the third quadrant. The angle that N_2 makes with the horizontal is:

$$\tan^{-1}\frac{2.07}{2.93} = \tan^{-1} .707 = 35°.05.$$

Hence:

$$N_2 = \frac{0.1\,E}{\cos 35^\circ} = 0.122\,E. \tag{563}$$

Its projection upon axis 2 is therefore:

$$J_2^{(2)} = -N_2 \sin 35^\circ = -0.0707\,E$$

which checks the value previously found.

These results are summarized in Fig. 86 in which both normal amplitudes N_1 and N_2 are drawn in their proper positions. We see that in this particular case the normal amplitudes are equal in length and make equal angles with the horizontal. This is merely a consequence of the particular values chosen for the network inductances and resistances and has no deeper significance. We see, however, that this determination of the normal coördinates and normal amplitudes unifies the situation regarding the transient mesh-current amplitudes for the various modes. It helps to clarify the significance of the transient solution.

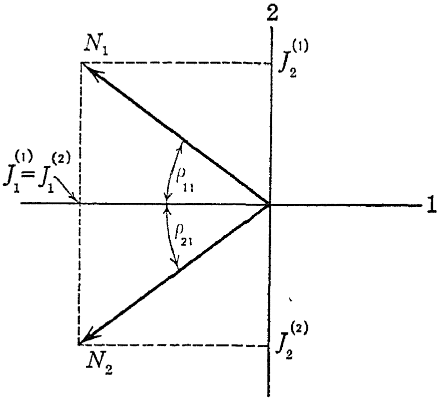


FIG. 86

From this simple illustration we note that the integration constants which we denote by $G^{(\nu)}$ are not the normal amplitudes directly, but are proportional to them, as mentioned above. Also the minors $C_{ik}^{(\nu)}$ are not the cosines of the angles between the directions of the normal amplitudes and the coördinate k directly, but are proportional to them. However, since the system of reference axes is orthogonal, we have:

$$(J_1^{(1)})^2 + (J_2^{(1)})^2 = (N_1)^2 \tag{564}$$

or:

$$(C_{i1}^{(1)}G^{(1)})^2 + (C_{i2}^{(1)}G^{(1)})^2 = (N_1)^2. \tag{564a}$$

Hence:

$$N_1 = G^{(1)}\sqrt{(C_{i1}^{(1)})^2 + (C_{i2}^{(1)})^2}. \tag{565}$$

Similarly:

$$N_2 = G^{(2)} \sqrt{(C_{i1}^{(2)})^2 + (C_{i2}^{(2)})^2}. \tag{566}$$

In general the normal amplitude will be determined by:

$$N_\nu = G^{(\nu)} \sqrt{(C_{i1}^{(\nu)})^2 + (C_{i2}^{(\nu)})^2 + \cdots + (C_{in}^{(\nu)})^2}. \tag{567}$$

If we denote the angle between the direction of the normal amplitude N_ν and the axis k by $\rho_{k\nu}$, then it follows that:

$$\cos \rho_{k\nu} = \frac{C_{ik}^{(\nu)}}{\sqrt{(C_{i1}^{(\nu)})^2 + (C_{i2}^{(\nu)})^2 + \cdots + (C_{in}^{(\nu)})^2}}. \tag{568}$$

From these considerations we then have:

$$J_k^{(\nu)} = N_\nu \cos \rho_{k\nu}. \tag{569}$$

From these equations N_ν always has the same algebraic sign as $G^{(\nu)}$, and $\cos \rho_{k\nu}$ has the sign of $C_{ik}^{(\nu)}$. If this is observed, then the algebraic signs of the J's will come out correct.

These formulae (567), (568), and (569) are given merely to round out the theory regarding normal coördinates and normal amplitudes. In the solution of a problem it is not necessary to determine either the normal amplitudes (567) or the direction cosines (568), except for the sake of interest.

It might be well to emphasize again, that when a pair of conjugate modes are considered, that pair together defines one normal coördinate. Since this pair of modes gives rise to a pair of conjugate exponential terms, the actual amplitude of this normal oscillation will be $2\,|N_\nu|$, where ν is the number of one of the modes. For such complex modes, N_ν by (567) will be complex, and so will $\cos \rho_{k\nu}$ by (568); i.e., both of these quantities will have a magnitude and an angle. The sum of these angles is the angle of the corresponding $J_k^{(\nu)}$, and determines the time phase of the component normal oscillation in mesh #k. This time phase is different in the different meshes as pointed out previously. If it were the same in all the meshes, then $\cos \rho_{k\nu}$ would be real and not complex. A complex cosine is, of course, hard to visualize. It means that when the normal amplitude N_ν is projected upon various reference coördinates, not only the magnitude of the oscillation, but also its time phase is affected.

7. Realization of Normal Meshes. From the above it appears as though these normal coördinates can only be realized physically in mechanical systems. In an electrical network they are in general mere fictitious meshes which are introduced into the picture because of the mathematical analogy between the mechanical and the electrical systems. The situation need not always be entirely fictitious, however. It is possible to so construct a network as to make one or more of the actual meshes become normal coördinates or normal meshes. We then say that these meshes have been **normalized.** Since a normal coördinate isolates its own mode, it follows that these normalized meshes will do the same. We shall now discuss how this may be done.

From the mechanical side, the possibility of coincidence between a normal coördinate and a reference coördinate is readily appreciated. We said above, that these normal coördinates or directions actually exist in mechanical systems. Suppose we have located one of these. Then we can orient our Cartesian reference axes so that one of them, say axis 1, coincides with the normal direction. Since the reference axes are all normal to each other, it then follows that this particular normal coördinate will be at right angles to all the axes except axis 1 with which it coincides. Hence, this normal oscillation can have no projections upon the other reference coördinates, and is therefore completely confined to axis 1.

If we should be able to make an actual mesh in an electrical network coincide with a normal mesh, then the corresponding normal mode of oscillation and decay would be confined to that normalized mesh. This may very easily be done. Consider the modular determinant (541), and suppose that we adjust the network parameters so that:

$$\left.\begin{aligned} c_{12} &= \eta_2 c_{11} \\ c_{13} &= \eta_3 c_{11} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_{1n} &= \eta_n c_{11} \end{aligned}\right\} \qquad (570)$$

where $\eta_2, \eta_3, \ldots \eta_n$ are proportionality constants. Then the determinantal equation would become:

$$D(p) = c_{11} \begin{vmatrix} 1 & c_{12} \ldots c_{1n} \\ \eta_2 & c_{22} \ldots c_{2n} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \eta_n & c_{n2} \ldots c_{nn} \end{vmatrix} = 0 \qquad (571)$$

because c_{11} could be factored out of the first row or column. But this means that the determinantal equation is factorable, and is represented by the two equations:

$$c_{11} = 0 \tag{572}$$

and:

$$\begin{vmatrix} 1 & c_{12} \dots c_{1n} \\ \eta_2 & c_{22} \dots c_{2n} \\ \dots & \dots \dots \\ \eta_n & c_{n2} \dots c_{nn} \end{vmatrix} = 0. \tag{573}$$

Equation (572) will in general be quadratic, and hence give rise to a pair of modes. (573) will determine the rest of the modes of the system. Suppose the pair of modes determined by (572) are conjugate complexes so that they define one frequency and one decrement. Let us denote this pair by $p_{1,2}$. The rest of the modes defined by (573) do not interest us at present.

Now consider the modular group of vectors for the modes $p_{1,2}$. Since:

$$\left.\begin{aligned} c_{11} &= 0 \\ c_{12} &= c_{21} = 0 \\ &\dots\dots \\ c_{1n} &= c_{n1} = 0 \end{aligned}\right\} \tag{574}$$

it follows that the vector c_1 vanishes altogether, and the rest of the vectors have no components on the reference axis 1. Hence, this axis is wholly unoccupied. The rank of the group is $n - 1$. The solution to the homogeneous system (539) is now evident. The desired position for the vector J is coincident with axis 1, i.e., coincident with mesh #1. This direction is, however, that of the normal coördinate for the rate of oscillation and decay defined by the modes $p_{1,2}$. Hence, mesh #1 becomes this normal coördinate.

The same idea may be carried out with respect to any other mesh in the network. We may make the kth mesh a normal coördinate by choosing the parameters such that:

$$\left.\begin{aligned} c_{k2} &= \eta_2 c_{kk} \\ c_{k3} &= \eta_3 c_{kk} \\ &\dots\dots \\ c_{kn} &= \eta_n c_{kk}. \end{aligned}\right\} \tag{575}$$

In order to see what the conditions (570) or (575) really mean we note that:

$$c_{ik} = L_{ik}p + R_{ik} + S_{ik}p^{-1}.$$

Therefore (575) requires that:

$$\left.\begin{aligned} L_{k2} &= \eta_2 L_{kk} \\ L_{k3} &= \eta_3 L_{kk} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ L_{kn} &= \eta_n L_{kk} \end{aligned}\right\} \tag{576}$$

plus the conditions:

$$\left.\begin{aligned} R_{k2} &= \eta_2 R_{kk} \\ R_{k3} &= \eta_3 R_{kk} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ R_{kn} &= \eta_n R_{kk} \end{aligned}\right\} \tag{577}$$

and:

$$\left.\begin{aligned} S_{k2} &= \eta_2 S_{kk} \\ S_{k3} &= \eta_3 S_{kk} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ S_{kn} &= \eta_n S_{kk}. \end{aligned}\right\} \tag{578}$$

In words, the ratios of the total inductance on the contour of mesh #*k* to that in any common branch between the *k*th mesh and any other to which it is coupled, must equal the same ratios for resistance and elastance. This requires that all the common branches which form part of the contour of mesh #*k* contain the same kinds of elements as the *k*th mesh contour. As an illustration consider the network of Fig. 87. Here mesh #1 contains resistance and inductance on its contour as well as in the common branch between 1 and 2. Hence mesh #1 may be made a normal coördinate. Mesh #2 could not be made a normal coördinate no matter how the constants were adjusted, because it contains elastance on its contour but not in the common branch.

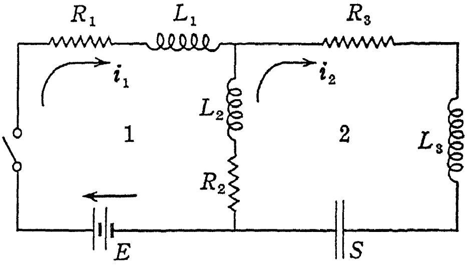


FIG. 87

In order to make mesh #1 a normal coördinate we must choose the constants R_1, R_2, L_1, and L_2 such that:

$$\frac{L_1 + L_2}{L_2} = \frac{R_1 + R_2}{R_2}. \tag{579}$$

This condition is identical with those expressed by (576) and (577) if we note that here $k = 1$ and only L_{k2} and R_{k2} need be considered because the network has only two meshes. Also:

$$\begin{aligned} L_{11} &= L_1 + L_2 \\ R_{11} &= R_1 + R_2 \\ L_{12} &= -L_2 \end{aligned}$$

and

$$R_{12} = -R_2.$$

Suppose we choose the following numerical values:

$$\left.\begin{aligned} L_1 &= 0.0015 \qquad & L_2 &= 0.001 \\ R_1 &= \quad 9 & R_2 &= \ 6. \end{aligned}\right\} \tag{580}$$

The rest of the constants may be fixed arbitrarily, thus:

$$L_3 = 0.0004 \qquad R_3 = 6.4 \qquad S = 10^9. \tag{581}$$

The network parameters become:

$$\left.\begin{aligned} L_{11} &= \ \ 0.0025 \qquad & R_{11} &= \ \ 15 \qquad & S_{11} &= 0 \\ L_{22} &= \ \ 0.0014 & R_{22} &= \ \ 12.4 & S_{22} &= 10^9 \\ L_{12} &= -0.001 & R_{12} &= -6.0 & S_{12} &= 0. \end{aligned}\right\} \tag{582}$$

Hence the elements of the modular determinant are:

$$\left.\begin{aligned} c_{11} &= 0.0025\,p + 15 \\ c_{22} &= 0.0014\,p + 12.4 + 10^9 p^{-1} \\ c_{12} &= -0.001\,p - 6 = c_{21}. \end{aligned}\right\} \tag{583}$$

We see that:

$$c_{12} = -0.4\,c_{11} \tag{584}$$

which makes mesh #1 a normal coördinate.

The complete solution proceeds as follows. By inspection we obtain the steady-state solutions:

$$\left.\begin{aligned} (i_1)_s &= 0.066\,E \\ (i_2)_s &= \quad 0. \end{aligned}\right\} \tag{585}$$

For the transient solutions we note that mesh #1 has a weight of one, and mesh #2 has a weight of two. Hence, the system has three modes, and the determinantal equation is of the third degree. In forming the latter, the modular determinant will have to be multiplied by p before setting it equal to zero. We have:

$$D(p) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11}c_{22} - c_{12}c_{21}$$

or:

$$D(p) = c_{11}(c_{22} - 0.16\, c_{11}).$$

Hence, the determinantal equation becomes:

$$c_{11}p(c_{22} - 0.16\, c_{11}) = 0 \tag{586}$$

or:

$$c_{11} = 0 \tag{586a}$$

and:

$$p(c_{22} - 0.16\, c_{11}) = 0. \tag{586b}$$

Equation (586*a*) determines the mode corresponding to the normal coördinate which coincides with mesh #1. Hence this is the isolated mode. Substituting from (583) into (586*a*), we have:

$$0.0025\, p + 15 = 0$$

which yields:

$$p_1 = -6000 \tag{587}$$

for the isolated mode.

Substituting from (583) into (586*b*), we get:

$$0.001\, p^2 + 10\, p + 10^9 = 0$$

which yields:

$$\left.\begin{aligned} p_2 &= (-5 + j\,1000)10^3 \\ \text{and:} \quad p_3 &= (-5 - j\,1000)10^3. \end{aligned}\right\} \tag{588}$$

The complete solutions, therefore, become:

$$\left.\begin{aligned} i_1 &= J_1^{(1)}e^{-6000\,t} + (J_1^{(2)}e^{j\,10^6 t} + J_1^{(3)}e^{-j\,10^6 t})e^{-5000\,t} + 0.066\,E \\ i_2 &= J_2^{(1)}e^{-6000\,t} + (J_2^{(2)}e^{j\,10^6 t} + J_2^{(3)}e^{-j\,10^6 t})e^{-5000\,t} \end{aligned}\right\} \tag{589}$$

or in terms of the true integration constants, we have:

$$\left.\begin{aligned} i_1 = {} & C_{11}^{(1)}G^{(1)}e^{-6000\,t} \\ & + (C_{11}^{(2)}G^{(2)}e^{j\,10^6 t} + C_{11}^{(3)}G^{(3)}e^{-j\,10^6 t})e^{-5000\,t} + 0.066\,E \\ i_2 = {} & C_{12}^{(1)}G^{(1)}e^{-6000\,t} \\ & + (C_{12}^{(2)}G^{(2)}e^{j\,10^6 t} + C_{12}^{(3)}G^{(3)}e^{-j\,10^6 t})e^{-5000\,t}. \end{aligned}\right\} \tag{590}$$

In order to be able to substitute the initial charge condition for mesh #2, we must determine the charge equation for this mesh by integrating the relation for i_2 in (590) with respect to time and adding the steady-state charge for this mesh. The latter is equal to the steady-state voltage on the contour of mesh #2 times the capacitance on that contour. This gives:

$$(q_2)_s = \frac{R_2 E}{(R_1 + R_2)S} = 4 \cdot 10^{-10}\, E \tag{591}$$

so that the charge equation for mesh #2 becomes:

$$\begin{aligned} q_2 = \frac{C_{12}{}^{(1)}G^{(1)}e^{-6000t}}{-6000} &+ \left(\frac{C_{12}{}^{(2)}G^{(2)} \cdot 10^{-3}e^{j\,10^6t}}{-5 + j\,1000} + \right. \\ &\left. \frac{C_{12}{}^{(3)}G^{(3)} \cdot 10^{-3}e^{-j\,10^6t}}{-5 - j\,1000}\right)e^{-5000t} + 4 \cdot 10^{-10}\, E. \end{aligned} \tag{592}$$

Let us now choose initial rest conditions, i.e.:

$$\left.\begin{aligned} q_{20} &= 0 \\ i_{10} &= 0 \\ i_{20} &= 0. \end{aligned}\right\} \tag{593}$$

Then we have by (590) and (592):

$$\left.\begin{aligned} &C_{11}{}^{(1)}G^{(1)} + C_{11}{}^{(2)}G^{(2)} + C_{11}{}^{(3)}G^{(3)} = -0.066E \\ &C_{12}{}^{(1)}G^{(1)} + C_{12}{}^{(2)}G^{(2)} + C_{12}{}^{(3)}G^{(3)} = 0 \\ &\frac{C_{12}{}^{(1)}G^{(1)}}{-6000} + \frac{C_{12}{}^{(2)}G^{(2)} \cdot 10^{-3}}{-5 + j\,1000} + \frac{C_{12}{}^{(3)}G^{(3)} \cdot 10^{-3}}{-5 - j\,1000} = -4 \cdot 10^{-10}\, E. \end{aligned}\right\} \tag{594}$$

We must now determine the values of the above minors. We have:

$$\left.\begin{aligned} C_{11}{}^{(\nu)} &= C_{22}{}^{(\nu)} = 0.0014\, p_\nu + 12.4 + 10^9 p_\nu{}^{-1} \\ C_{12}{}^{(\nu)} &= -C_{21}{}^{(\nu)} = 0.001\, p_\nu + 6. \end{aligned}\right\} \tag{595}$$

Hence:

$$\left.\begin{aligned} &C_{11}{}^{(1)} = -1.66 \cdot 10^5 \quad C_{11}{}^{(2)} = 0.4 + j\,400 \quad C_{11}{}^{(3)} = 0.4 - j\,400 \\ &C_{12}{}^{(1)} = 0 \qquad\qquad\quad C_{12}{}^{(2)} = 1 + j\,1000 \quad C_{12}{}^{(3)} = 1 - j\,1000. \end{aligned}\right\} \tag{596}$$

The equations (594) therefore become:

$$\left.\begin{aligned} &-1.66 \cdot 10^5 G^{(1)} \\ &\quad + \quad (0.4 + j\,400)G^{(2)} + (0.4 - j\,400)G^{(3)} = -0.066\, E \\ &\ 0 + (1 + j\,1000)G^{(2)} + (1 - j\,1000)G^{(3)} = 0 \\ &\ 0 + (10^{-3} - j\,6 \cdot 10^{-6})G^{(2)} + (10^{-3} + j\,6 \cdot 10^{-6})G^{(3)} = -4 \cdot 10^{-10} E. \end{aligned}\right\} \tag{597}$$

The solutions are:

$$\left.\begin{aligned} G^{(1)} &= 4 \cdot 10^{-7}\,E \\ G^{(2)} &= -2(1 + j\,10^{-3})10^{-7}\,E \\ G^{(3)} &= -2(1 - j\,10^{-3})10^{-7}E. \end{aligned}\right\} \qquad (598)$$

Hence the solutions (590) become:

$$\left.\begin{aligned} i_1 &= 0.066\,E(1 - e^{-6000t}) + 1.6 \cdot 10^{-4} Ee^{-5000t} \sin 10^6 t \\ i_2 &= 4 \cdot 10^{-4}\,Ee^{-5000t} \sin 10^6\,t. \end{aligned}\right\} \qquad (599)$$

We see that the decrement $p_1 = -6000$ is absent in the second mesh. This is to be expected, since the modular vector group for this mode is given by the components:

$$\begin{aligned} c_{11}{}^{(1)} &= 0 \qquad & c_{12}{}^{(1)} &= 0 \\ c_{21}{}^{(1)} &= 0 \qquad & c_{22}{}^{(1)} &= -1.66 \cdot 10^5 \end{aligned} \qquad (600)$$

as may be verified by substituting this mode into the relations (583). One of these vectors is zero altogether, and the other one lies along the axis 2. Hence, axis 1 becomes the normal coördinate for this mode. It is at right angles to axis 2, and hence has no projection upon it. This is why the mode p_1 is absent in mesh #2. Incidentally the direct method of evaluating the integration constants is in this case preferable as compared to the one by means of successive initial derivatives.

8. Interpretation of Network Phenomena in Terms of the Excitation of Normal Coördinates. The preceding discussion of the geometrical significance of normal coördinates leads us to a new interpretation of the excitation phenomena in a network. We are already familiar with the fact that when a disturbance occurs in any mesh of a network, the entire network responds. If the network possesses m modes, then in general the response in every mesh will consist of a linear superposition of m normal functions with these modes as exponents. This may be given a geometric interpretation.

Consider a two-mesh network with resistance, inductance, and elastance in each mesh as well as in the common branch. If the relative magnitudes of the parameters are such that the modes are complex, then these will come in two conjugate pairs, and will define two natural frequencies with corresponding decrements, and hence two normal coördinates. This is represented geometrically in Fig. 88, where N_1 and N_2 are the two normal coördinates, and the axes 1 and 2 represent the two physical meshes.

Suppose now that a disturbance, such as the sudden insertion of a voltage, occurs in mesh #1, i.e., in axis 1. From the figure we see that this axis has projections upon both normal coördinates; i.e., the motion of a point along axis 1 would cause projected motions upon both of the normal axes N_1 and N_2. Hence, we see that both pairs of conjugate modes are excited by the disturbance along axis 1. But since both normal coördinates also have projections upon axis 2, these same modes of oscillation and decay will also appear in axis 2, i.e., in mesh #2, with relative amplitudes governed by the direction cosines for the normal coördinates. Hence, we see that the excitation is transmitted from axis 1 to axis 2 via the normal coördinates, and that is why both normal modes in general appear in both meshes. For networks with more meshes and more normal coördinates, the same general argument holds.

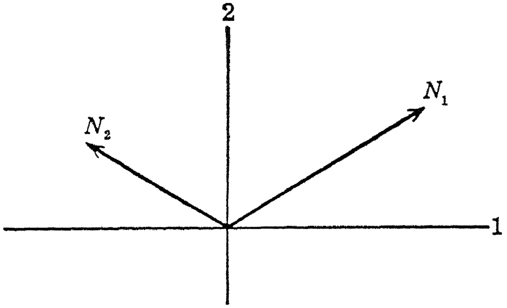


FIG. 88

This interpretation of the transmission of excitation from one mesh to all the others is more fundamental than to say that it is transmitted by means of the mutual parameters. For example, in the above numerical illustration we saw that the disturbance in mesh #1 of the network of Fig. 87 excited both meshes, but only the pair of conjugate modes p_2 and p_3 appeared in mesh #2. This was due to the fact that the normal coördinate N_1 in this case coincided with axis 1, so that — although it was excited by the disturbance in mesh #1 — it possessed no projection upon mesh #2, and hence could not transmit its mode to that mesh.

This line of reasoning leads us to predict another peculiarity of the network of Fig. 87. Let us explain this on the basis of Fig. 89 which illustrates the geometry of the situation. The absolute lengths of the normal coördinates are not drawn to scale. They are merely shown in their relative angular positions. The normal coördinate N_1 corresponds to the mode $p_1 = -6000$, while the coördinate N_2 corresponds to the pair of conjugate modes $p_{2,3} = -5000 \pm j\,10^6$ which give rise to the damped sinusoids in the solutions (599). The angular position of this latter coördinate is such that its projection upon axis 2 is 2.5 times that on axis 1, which makes the amplitude of the sine term in i_2 2.5 times that in i_1.

Suppose we shifted the battery and switch of the network of Fig. 87 from mesh #1 to mesh #2. The excitation would then originate in mesh #2. From Fig. 89 we see that a disturbance along axis 2 possesses no projection upon N_1. Hence the corresponding mode p_1 cannot be excited. The oscillation and decay corresponding to N_2 are excited and transmitted to axis 1 via their normal coördinate. This is a very interesting result, namely that the mode p_1 does not come into action at all, not even in its own coördinate. Hence, we see that when a normal coördinate coincides with an actual mesh in a network, then it can be excited only by a disturbance in that mesh, and in that case its action is confined entirely to that mesh.

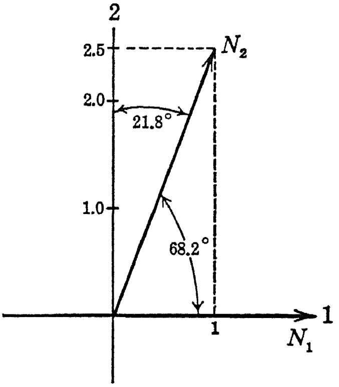


FIG. 89

We can show numerically that the above prediction is correct. When the battery is placed in mesh #2 of the network of Fig. 87, then the steady-state currents in both meshes become zero while the steady-state charge in mesh #2 becomes:

$$10^{-9}\,E \tag{601}$$

instead of the value given by (591). Hence the equations (597) for the evaluation of the integration constants become:

$$\left.\begin{array}{l} -1.66\cdot 10^{5}G^{(1)} \\ \qquad + \quad (0.4+j\,400)G^{(2)} + \quad (0.4-j\,400)G^{(3)} = 0 \\ \quad 0 + \quad (1+j\,1000)G^{(2)} + \quad (1-j\,1000)G^{(3)} = 0 \\ \quad 0 + (10^{-3}-j\,6\cdot 10^{-6})G^{(2)} + (10^{-3}+j\,6\cdot 10^{-6})G^{(3)} = -10^{-9}E. \end{array}\right\} \tag{602}$$

The solutions are:

$$\left.\begin{aligned} G^{(1)} &= 0 \\ G^{(2)} &= -5\,(1 + j\,10^{-3}) \cdot 10^{-7}\,E \\ G^{(3)} &= -5\,(1 - j\,10^{-3}) \cdot 10^{-7}\,E. \end{aligned}\right\} \tag{603}$$

Since $G^{(1)}$ is zero, there will be no term with the mode p_1 in either mesh. For the other modes we have:

$$\left.\begin{aligned} J_1^{(2)} &= C_{11}^{(2)}G^{(2)} = -2j\,10^{-4}\,E \\ J_1^{(3)} &= C_{11}^{(3)}G^{(3)} = +2j\,10^{-4}\,E \end{aligned}\right\} \tag{604}$$

and:

$$\left.\begin{aligned} J_2^{(2)} &= C_{12}^{(2)}G^{(2)} = -5j\,10^{-4}\,E \\ J_2^{(3)} &= C_{12}^{(3)}G^{(3)} = +5j\,10^{-4}\,E. \end{aligned}\right\} \tag{605}$$

The complete solutions are therefore:

$$\left.\begin{aligned} i_1 &= 4 \cdot 10^{-4}\,Ee^{-5000\,t}\sin 10^6\,t \\ i_2 &= 10^{-3}\,Ee^{-5000\,t}\sin 10^6\,t. \end{aligned}\right\} \tag{606}$$

We see that i_2 is 2.5 times i_1 as we had predicted, and that the exponential decay corresponding to p_1 is absent altogether. It is further interesting to note that i_1 for this case is identical with i_2 of (599) for the impressed voltage in mesh #1. This is due to the reciprocal theorem. We mentioned this theorem in Chapter IV in connection with the steady-state performance of a network, and also demonstrated its correctness for the steady state by the determinant method. We have, however, not yet seen the reciprocal theorem in connection with the transient behavior except in the present instance. It does hold for the transient solution provided the network is initially at rest as was assumed in the above problem. Hence, it holds for the complete solution under this provision. Its proof would be rather cumbersome to carry out in terms of the preceding evaluation processes, and we, therefore, omit it here. It may be proven very elegantly in connection with Heaviside's expansion formula for the evaluation of integration constants. We merely call attention to the reciprocal theorem here because it is so evident in the above example, and therefore naturally excites interest.

The above example incidentally illustrates the geometrical significance of a normal coördinate for a pair of conjugate modes. At the close of section 5 of this chapter we pointed out that N, by formula (567) and cos $\rho_{k\nu}$ by (568) were complex when determined for a pair of complex modes. A complex length and a complex angle cannot, of course, be plotted as N_2 is plotted for this case in Fig. 89. This figure, however, represents only the magnitude of cos $\rho_{k\nu}$. In other words, when complex modes are considered and it is desired to represent the normal coördinates geometrically, only their magnitudes should be taken into account. If the magnitude of N_ν is plotted, then the amplitude of oscillation in any axis is twice its projection upon that axis. If $2\,|N_\nu|$ is plotted, then the projections are the amplitudes of oscillation themselves. The factor of *two* comes in due to the fact that two conjugate complex terms make one oscillatory real term. The angles of the complex N_ν and cos $\rho_{k\nu}$ are not angles in the Fig. 89, but are time-phase angles.

For example, using (596) and (598) in the above case we have for N_2:

$$\begin{aligned} N_2 &= G^{(2)}\sqrt{(C_{11}{}^{(2)})^2 + (C_{12}{}^{(2)})^2} \\ &= -2\,(1 + j\,10^{-3}) \cdot 10^{-7}\,E \cdot (1 + j\,10^3) \cdot 1.078 \\ &= -j\,2.156\,E \cdot 10^{-4} \end{aligned} \tag{607}$$

and for the cosine for mesh #1 and mode #2 we have:

$$\begin{aligned} \cos\rho_{12} &= \frac{C_{11}{}^{(2)}}{\sqrt{(C_{11}{}^{(2)})^2 + (C_{12}{}^{(2)})^2}} \\ &= \frac{0.4 + j\,400}{(1 + j\,1000) \cdot 1.078} \\ &= 0.372. \end{aligned} \tag{608}$$

The cosine happens to come out real in this case. The corresponding angle that the normal coördinate N_2 makes with axis 1 is:

$$\cos^{-1} 0.372 = 68.2° \tag{609}$$

which is indicated in Fig. 89. If (608) had come out complex, only the magnitude would have been considered. The fact that it is real means that the time phase of the corresponding oscillation is the same in both meshes. This checks with the above results (599) and (606), which indicate a sine term with zero phase for both meshes.

9. Resonance Phenomena in Terms of the Excitation of Normal Coördinates. In general we have seen that resonance phenomena occur whenever an impressed frequency coincides with a natural frequency of the network, i.e., with the imaginary portion of a pair of conjugate complex modes. When the network has no normalized meshes, then all of the modes, and hence all of the natural frequencies, may appear in all of the meshes. When a steady-state frequency, equal to one of the natural frequencies, is impressed in any mesh of the network, it excites an exceptional amplitude in the corresponding fictitious normal mesh, and through it excites exceptional amplitudes in all of the actual meshes in the network. In this respect the excitation phenomena in the steady state act like those for the transient state. The concept of normal coördinates and their projections upon the actual physical coördinates of the system, is valuable in the interpretation of steady-state phenomena as well as of the natural behavior.

Resonance phenomena can, of course, be justified mathematically on the basis of an investigation of the steady-state solution:

$$I_k = \frac{E_i B_{ik}}{D} \tag{610}$$

given by eq. (327) of Chapter IV. Here D is the network determinant and is a function of the impressed angular frequency ω. It differs from the modular determinant $D(p)$ in that p is replaced by $j\omega$. For this reason it is also frequently written $D(j\omega)$. Since the modes are determined from the relation:

$$D(p) = 0$$

it follows that whenever ω equals the imaginary portion of a complex mode:

$$D(j\omega) = \text{minimum}. \tag{611}$$

It will not go to zero at this point because the impressed mode $(j\omega)$ is not wholly equal to the natural mode in question, but only to its imaginary part. The minor B_{ik} in the numerator of the expression (610) is, of course, also a function of ω, but in general it will vary relatively slowly in the region where $D(j\omega)$ becomes a minimum. Hence, the maxima of the function (610) occur practically where the minima of $D(j\omega)$ occur. The number of these minima are obviously equal to the number of pairs of conjugate modes. Thus, the steady-state resonance phenomena may be investigated from a study of the relation (611) together with (610).

In general the picture becomes clearer, however, when analyzed on the basis of the normal coördinates. This is especially true when the network possesses normalized meshes. The above numerical example for the network of Fig. 87 cannot be used to illustrate this point because the mode belonging to the coincident normal coördinate N_1 is not oscillatory, i.e., has no imaginary part. Let us, therefore, consider the network of Fig. 90 which contains resistance, inductance, and elastance in both meshes as well as in the common branch. E_1 is the root-mean-square amplitude of the impressed harmonic voltage having an angular frequency of ω radians per second. The constants have the following values:

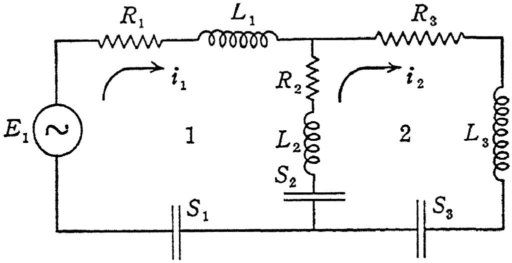


FIG. 90

$$\left.\begin{array}{lll} R_1 = 4 & R_2 = 6 & R_3 = 7.6 \\ L_1 = .0004 & L_2 = .0006 & L_3 = .00176 \\ S_1 = 0.4 \cdot 10^9 & S_2 = 0.6 \cdot 10^9 & S_3 = 0.76 \cdot 10^9. \end{array}\right\} \quad (612)$$

Thus the network parameters become:

$$\left.\begin{array}{lll} L_{11} = .001 & R_{11} = 10 & S_{11} = 10^9 \\ L_{12} = -.0006 & R_{12} = -6 & S_{12} = -0.6 \cdot 10^9 \\ L_{22} = .00236 & R_{22} = 13.6 & S_{22} = 1.36 \cdot 10^9. \end{array}\right\} \quad (613)$$

The elements of the modular determinant are:

$$\left.\begin{array}{l} c_{11} = .001\,p + 10 + 10^9\,p^{-1} \\ c_{12} = -0.6\,c_{11} = c_{21} \\ c_{22} = .00236\,p + 13.6 + 1.36 \cdot 10^9\,p^{-1}. \end{array}\right\} \quad (614)$$

The modes are determined from:

$$p^2 D(p) = 0$$

or:

$$p^2 c_{11}(c_{22} - 0.36\,c_{11}) = 0. \quad (615)$$

Mesh #1 is again a normal coördinate. The corresponding mode is determined from:

$$pc_{11} = .001\,p^2 + 10\,p + 10^9 = 0 \tag{615a}$$

which yields:

$$\left.\begin{aligned} p_1 &= -5000 + j\,10^6 \\ p_2 &= -5000 - j\,10^6. \end{aligned}\right\} \tag{616}$$

The other pair of modes is determined from:

$$p(c_{22} - 0.36\,c_{11}) = .002\,p^2 + 10\,p + 10^9 = 0 \tag{615b}$$

which yields:

$$\left.\begin{aligned} p_3 &= -2500 + j\,0.707 \cdot 10^6 \\ p_4 &= -2500 - j\,0.707 \cdot 10^6. \end{aligned}\right\} \tag{617}$$

The components of the modular vector group for p_1 are given by:

$$\left.\begin{aligned} c_{11} &= 0 \qquad & c_{12} &= 0 \\ c_{21} &= 0 \qquad & c_{22} &= -5 + j\,1000. \end{aligned}\right\} \tag{618}$$

The components of the modular vector group for p_3 are given by:

$$\left.\begin{aligned} c_{11} &= 5 - j\,293 \qquad & c_{12} &= -0.6\,c_{11} \\ c_{21} &= -0.6\,c_{11} \qquad & c_{22} &= 0.36\,c_{11}. \end{aligned}\right\} \tag{619}$$

These vector groups together with the corresponding normal coördinates are plotted in Fig. 91 (*a*) and (*b*) respectively. The normal amplitudes are denoted by $N_{1,2}$ and $N_{3,4}$ for the pairs of modes p_1, p_2 and p_3, p_4, respectively. They are not drawn to scale as to length because the latter is determined by the initial conditions about which we have as yet said nothing at all. We are at present

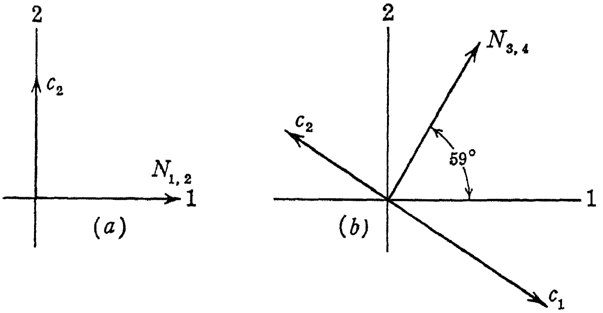


FIG. 91

only interested in the angular positions of the normal coördinates with respect to the axes, i.e., the physical meshes in the network.

$N_{1,2}$ coincides with axis 1 as it should. $N_{3,4}$ makes an angle of 59° with the horizontal. Hence for a given amplitude it will have a larger projection upon axis 2 than upon axis 1, this ratio being 1.66. With this in mind we can predict the following, relative to the resonance phenomena. With the generator in mesh #1, resonance will occur for two frequencies, namely at $\omega = 0.707 \cdot 10^6$ and again at $\omega = 10^6$. The former belongs to $N_{3,4}$, and hence will produce exceptional amplitudes in both meshes, the ratio of that in mesh #2 to that in mesh #1 being 1.66. The second resonance at 10^6 will be confined to mesh #1 alone, i.e., mesh #2 will not respond. With the generator in mesh #2, resonance will occur only at $\omega = 0.707 \cdot 10^6$, because the normal coördinate $N_{1,2}$ cannot be excited from axis 2. The single resonance at $0.707 \cdot 10^6$ will have amplitudes in meshes 2 and 1 in the ratio of 1.66 to 1 respectively.

We close this discussion with a numerical calculation of the above resonance phenomena to serve as a check on our prediction. For this purpose we must first determine the elements of the network determinant. These are:

$$\left.\begin{aligned} b_{11} &= .001\,j\omega + 10 + \frac{10^9}{j\omega} \\ b_{12} &= -0.6\,b_{11} = b_{21} \\ b_{22} &= .00236\,j\omega + 13.6 + \frac{1.36 \cdot 10^9}{j\omega}. \end{aligned}\right\} \qquad (620)$$

With the generator in mesh #1 we have:

$$\left.\begin{aligned} I_1 &= \frac{B_{11}E}{D} \\ I_2 &= \frac{B_{12}E}{D}. \end{aligned}\right\} \qquad (621)$$

But:

$$D = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}(b_{22} - 0.36\,b_{11}).$$

Also:

and:

$$\left.\begin{aligned} B_{11} &= b_{22} \\ B_{12} &= -b_{21} = 0.6\,b_{11}. \end{aligned}\right\} \qquad (622)$$

Hence:

$$\left.\begin{aligned} I_1 &= \frac{b_{22}E}{b_{11}(b_{22} - 0.36\,b_{11})} \\ I_2 &= \frac{0.6\,E}{b_{22} - 0.36\,b_{11}} \end{aligned}\right\} \tag{623}$$

which become:

$$I_1 = \frac{\left\{13.6 + j\left(.00236\,\omega - \dfrac{1.36 \cdot 10^9}{\omega}\right)\right\}E}{\left\{10 + j\left(.001\,\omega - \dfrac{10^9}{\omega}\right)\right\}\left\{10 + j\left(.002\,\omega - \dfrac{10^9}{\omega}\right)\right\}} \tag{624}$$

and:

$$I_2 = \frac{0.6\,E}{10 + j\left(.002\,\omega - \dfrac{10^9}{\omega}\right)}\,. \tag{625}$$

The magnitudes of these are shown plotted against ω in Fig. 92. We see that I_1 has two maxima, one at $\omega = 0.707 \cdot 10^6$ and one at $\omega = 10^6$. I_2 has only one maximum at $\omega = 0.707 \cdot 10^6$ which is 1.66 times the maximum of I_1 at this point. These results substantiate what was said above.

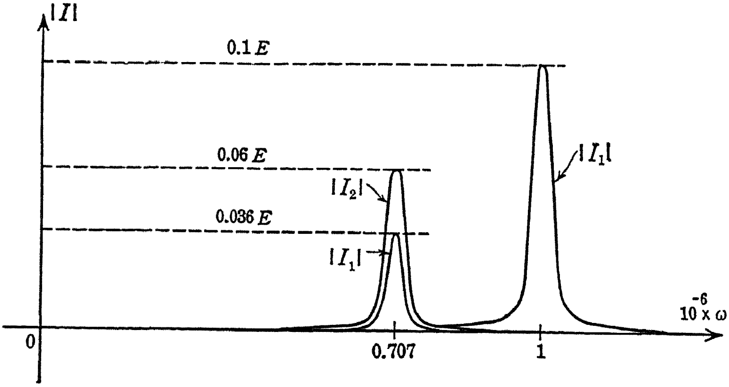


FIG. 92

Finally, with the generator in mesh #2 we have:

$$\left.\begin{aligned} I_1 &= \frac{B_{21}E}{D} \\ I_2 &= \frac{B_{22}E}{D}\,. \end{aligned}\right\} \tag{626}$$

But:

and:

$$\left.\begin{aligned} B_{21} &= -b_{12} = 0.6\,b_{11} \\ B_{22} &= b_{11}. \end{aligned}\right\} \tag{627}$$

Hence:

$$\left.\begin{aligned} I_1 &= \frac{0.6\,E}{b_{22} - 0.36\,b_{11}} \\ I_2 &= \frac{E}{b_{22} - 0.36\,b_{11}}\,. \end{aligned}\right\} \tag{628}$$

The magnitudes of these expressions are shown plotted against ω in Fig. 93. The curve for I_1 here corresponds to that for I_2 in Fig. 92 as it should. Resonance in both meshes occurs only for $\omega = 0.707 \cdot 10^6$, and the maximum for I_2 is 1.66 times that for I_1 as predicted above.

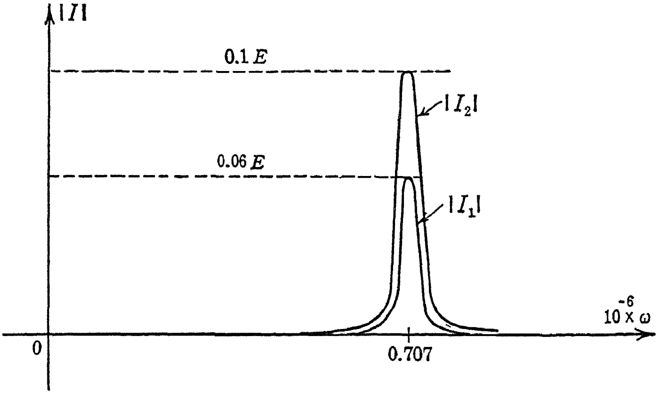


FIG. 93

PROBLEMS TO CHAPTER VII

7–1. Determine the geometric positions of the normal coördinates for the network of Prob. **6–7.**

7–2. On the basis of the results of Prob. **7–1,** what can you predict in regard to the relative magnitudes of the resonance maxima in the various meshes? Check your predictions by calculation.

7–3. On the basis of the results of Prob. **7–1,** check the relative magnitudes of the transient terms in the Prob. **6–7.**

7–4. A three-mesh network consists of a chain of identical meshes coupled by the same value of purely mutual inductance. Each mesh contains only resistance and inductance. Determine the geometric positions of the normal coordinates, and show that these form an orthogonal group.

7-5. On the basis of the results of Prob. **7-4,** predict the relative magnitudes of the transient terms in the various meshes.

7-6. Check the predictions of Prob. **7-5** for a constant force suddenly impressed in: *(a)* The first mesh; *(b)* the second mesh; *(c)* the third mesh.

7-7. A two-mesh network is characterized by the following parameters:

$$\begin{array}{lll} R_{11} = 20 & R_{12} = 10 & R_{22} = 12.5 \\ L_{11} = \dfrac{1}{\pi^2} & L_{12} = \dfrac{1}{2\,\pi^2} & L_{22} = \dfrac{5}{8\,\pi^2} \\ S_{11} = 2 \cdot 10^6 & S_{12} = 10^6 & S_{22} = 2 \cdot 10^6. \end{array}$$

Construct the network. Determine the geometrical positions of the normal coördinates, and predict the relative magnitudes of the normal functions in the two meshes, as well as of the steady-state amplitudes.

7-8. Determine the steady-state amplitudes of the currents in the network of Prob. **7-7,** and plot these as functions of frequency for the range:

$$2\,\pi \cdot 500 < \omega < 2\,\pi \cdot 1500,$$

(*a*) For a generator impressed in mesh #1;

(*b*) For a generator impressed in mesh #2.

7-9. Determine the transient solution for the network of Prob. **7-7,** and evaluate the integration constants by the direct method.

7-10. Repeat Prob. **7-9** using the method of successive initial derivatives, and compare.

CHAPTER VIII

APPLICATIONS

1. The Steady-State Solution to the Transformer Problem. The transformer is probably the most common piece of apparatus used in connection with communication networks. We will discuss here some of its outstanding characteristics. We shall not take up the non-linear effect produced by the introduction of an iron core, but will confine ourselves to the consideration of the air-core transformer only. The results so obtained also cover the general behavior of iron-core transformers. The latter have, of course, additional peculiarities which are much more complicated to investigate. We are at present not adequately prepared to take up these points. At all events, the characteristics of the air-core transformer should be thoroughly studied before the modifications due to the iron are considered.

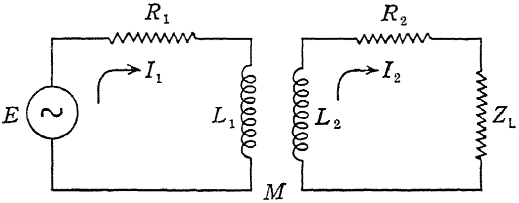


FIG. 94

A transformer, together with its source of excitation and simple load, forms essentially a two-mesh network. We shall study the steady-state behavior of this network here. It is illustrated in Fig. 94. R_1 and L_1 are the resistance and inductance respectively of the primary, while R_2 and L_2 are those constants for the secondary. M is the mutual inductance between the coils, and Z_L represents a load impedance whose structure we leave unspecified for the present. E is the root-mean-square amplitude of the applied voltage and may be due to any kind of a source. It is assumed to be harmonic with an angular frequency of ω radians per second.

The root-mean-square current amplitudes in the primary and secondary are denoted by I_1 and I_2 respectively.

If we let:

$$\left.\begin{aligned} b_{11} &= R_1 + jL_1\omega \\ b_{12} &= jM\omega = b_{21} \\ b_{22} &= R_2 + jL_2\omega + Z_L \end{aligned}\right\} \qquad (629)$$

then the time-average equilibrium conditions become:

$$\left.\begin{aligned} b_{11}I_1 + b_{12}I_2 &= E \\ b_{21}I_1 + b_{22}I_2 &= 0. \end{aligned}\right\} \qquad (630)$$

Their solution is given by:

$$\left.\begin{aligned} I_1 &= \frac{b_{22}E}{b_{11}b_{22} - b_{12}^2} \\ \text{and:} \qquad I_2 &= \frac{-b_{12}E}{b_{11}b_{22} - b_{12}^2}. \end{aligned}\right\} \qquad (631)$$

Before proceeding with the discussion of these results, let us reflect for a moment as to what kind of an apparatus a transformer is supposed to be, i.e., what function we usually expect it to perform. As its name implies it transforms something, usually the voltage and current as viewed either from the primary or secondary. We may also look upon it as a mechanism for transforming an impedance. This is a commonly accepted function of a transformer in communication work. The desired transformation, i.e., the ideal transforming property, is one of simple multiplication or division. There may be special cases where other characteristics are desired, but the usual object of using a transformer is to make a given impedance look a^2 times larger or smaller, where a^2 is a constant, but not necessarily a real number. In communication work we further require that the transformer shall do this not only at one frequency, but at all frequencies within a continuous band, the limits of which are fixed by certain standards of intelligibility and quality.

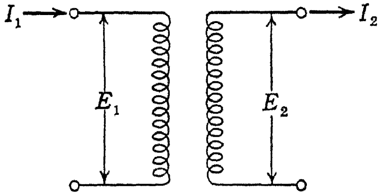


FIG. 95

These are the ideal characteristics of a transformer. It is our problem to investigate the coupled circuit of Fig. 94, and see how closely the ideal characteristics may be approximated by properly adjusting the vital parameters.

The ideal characteristics are made more evident by the Fig. 95 which illustrates what is supposed to be an ideal transformer. The input voltage and current are denoted by E_1 and I_1, while the output quantities are denoted by E_2 and I_2. The transformer is said to be ideal if:

and:

$$\left.\begin{aligned} E_1 &= aE_2 \\ I_1 &= -\frac{I_2}{a} \end{aligned}\right\} \tag{632}$$

where a is called the ratio of transformation. Obviously, if E_2 is considered as the voltage impressed upon a load impedance Z_L, then:

and:

$$\left.\begin{aligned} Z_L &= -\frac{E_2}{I_2} \\ Z_L' &= \frac{E_1}{I_1} = a^2 Z_L \end{aligned}\right\} \tag{633}$$

where Z_L' is the load impedance as viewed from the primary.

Now consider the solutions (631). From these we can form:

$$\frac{I_1}{I_2} = -\frac{b_{22}}{b_{12}}. \tag{634}$$

Substituting from (629) into the above, we have:

$$\frac{I_1}{I_2} = -\frac{R_2 + jL_2\omega + Z_L}{jM\omega}. \tag{635}$$

If the circuit is to act like an ideal transformer, then this expression must be a constant, i.e., independent of frequency. Obviously, it is not. However, if the term $L_2\omega$ were made very much larger than either R_2 or Z_L, i.e., if:

$$\left.\begin{aligned} L_2\omega &\gg R_2 \\ L_2\omega &\gg Z_L \end{aligned}\right\} \tag{636}$$

at all the essential frequencies, then (635) would become, with sufficient accuracy:

$$\frac{I_1}{I_2} = -\frac{L_2}{M} \tag{637}$$

which is a constant as required. In order to interpret the significance of (637) we recall that:

and:
$$\left.\begin{aligned} L_{22} &= k_{22}n_2^2 \\ M &= k_{12}n_1n_2 \end{aligned}\right\} \tag{638}$$

where k_{22} is substantially the permeance of the path which the flux must take in linking the secondary turns n_2, and k_{12} is substantially the permeance of the common flux path linking the primary turns n_1 and the secondary turns n_2, both expressed in practical units, of course. Then (637) becomes:

$$\frac{I_1}{I_2} = -\frac{k_{22}n_2^2}{k_{12}n_1n_2} \backsimeq -\frac{n_2}{n_1} \tag{639}$$

which is the inverse ratio of turns. The minus sign has no significance since the algebraic sign of M will depend upon the relative winding senses of the primary and secondary, of which we have said nothing. We are not interested in this anyway. We merely wish to point out that the result (637), based upon the assumption (636), leads to what is substantially the inverse ratio of turns of the primary and secondary windings.

Let us next see what condition yields the proper ratio of voltages demanded by (632). If we define E_2 as an impressed voltage at the load, then:

$$E_2 = -I_2Z_L. \tag{640}$$

Our voltage E in the solutions (631) may be denoted by E_1 to distinguish it as a primary voltage. Then by the second equation (631) we have:

$$\frac{E_1}{E_2} = \frac{b_{11}b_{22} - b_{12}^2}{Z_Lb_{12}}. \tag{641}$$

Substituting from (629) into the above, we obtain:

$$\frac{E_1}{E_2} = \frac{(M^2-L_1L_2)\omega^2+j(R_1L_2+R_2L_1)\omega+jL_1\omega Z_L+R_1(R_2+Z_L)}{jM\omega Z_L}. \tag{642}$$

The conditions under which this becomes substantially a constant are:

$$\left.\begin{aligned} Z_L \cdot L_1\omega &\gg (M^2 - L_1L_2)\omega^2 \\ Z_L \cdot L_1\omega &\gg (R_1L_2 + R_2L_1)\omega \\ Z_L \cdot L_1\omega &\gg R_1(R_2 + Z_L). \end{aligned}\right\} \tag{643}$$

Then (642) becomes very nearly:

$$\frac{E_1}{E_2} = \frac{L_1}{M}. \tag{644}$$

If we recall that:

$$L_1 = k_{11}n_1^2 \tag{645}$$

then (644) may be written:

$$\frac{E_1}{E_2} = \frac{k_{11}n_1^2}{k_{12}n_1n_2} \backsimeq \frac{n_1}{n_2} \tag{646}$$

which is the desired ratio of primary to secondary turns.

The significance of the first condition (643) is rather interesting. If we note equations (638) and (645), we see that:

$$M^2 - L_1L_2 = n_1^2n_2^2(k_{12}^2 - k_{11}k_{22}). \tag{647}$$

This expression will vanish if:

$$k_{11} = k_{22} = k_{12}. \tag{648}$$

This requires that the flux paths for the primary, the secondary, and the mutual turns be identical, which is the same as to say that there be no leakage flux. With an iron-core transformer this is more nearly the case than with an air-core transformer. Incidentally, when the leakage is negligible, then the approximate equalities in the expressions (639) and (646) become exact.

We can summarize the requirements of an ideal transformer, according to the conditions (636) and (643), as follows. For the essential frequency range the transformer must be so designed that:

> Its primary and secondary resistances are negligible compared to its inductive reactances; its magnetic leakage is negligible; and its inductive reactances are large compared to the impedance of the load.

So long as these conditions are complied with, the transformer will behave merely as a multiplying or dividing agent for the load impedance. It is not a simple matter to meet these requirements in practice. For audio frequencies the leakage is kept low by the use of iron and by proper physical design. The use of iron also helps the first and last requirements. The latter, of course, depends upon the nature of the load, so that this is a vital factor in determining the behavior of the transformer.

Note also that the conditions (636) for the current ratio and (643) for the voltage ratio contradict each other regarding the magnitude of the load impedance. If the latter is relatively small, then the current ratio will be more nearly ideal, but the voltage ratio will be farther from ideal, and vice versa. This is due to the fact that the current ratio will be nearly inversely as the turns when the load current is large compared to the exciting current in the transformer, while the voltage ratio will be nearly equal to the turns ratio at no load, i.e., with the secondary open.

Hence, we see that an actual transformer may act approximately as an impedance multiplier over a limited range of frequencies provided it is properly designed for the specific load in question. The more common use for a transformer, however, is to transform the voltage impressed upon its terminals. It is used in this capacity when placed between the output of one vacuum tube and the input of another. In such a case we are not interested in the current ratio at all, but only in the voltage ratio. Our object is to have the latter remain as constant as possible over the essential frequency range. Although the attainment of this end is quite difficult also, it is not as involved as the above problem. We shall consider this in detail now by carrying through the computations for several typical cases.

Let us consider in each case the ratio of the load voltage to the primary voltage. For this purpose the expression (642) is more useful in the following form:

$$\frac{E_2}{E_1} = \frac{jM\omega Z_L}{[R_1(R_2+Z_L)-(L_1L_2-M^2)\omega^2]+j\omega[R_1L_2+(R_2+Z_L)L_1]}. \qquad (649)$$

A pure resistance load is the commonest. For this we put:

$$Z_L = R. \qquad (650)$$

Then if we let:

$$\left.\begin{aligned} a &= R_1(R_2 + R) \\ b &= L_1L_2 - M^2 \\ d &= R_1L_2 + R_2L_1 + RL_1 \end{aligned}\right\} \qquad (651)$$

we have

$$\frac{E_2}{E_1} = \frac{jMR\omega}{(a - b\omega^2) + jd\omega}. \qquad (652)$$

In order to facilitate the discussion of this result it is convenient to introduce the notation:

$$\omega_1 = \sqrt{\frac{a}{b}} \tag{653}$$

and:

$$\omega_2 = \frac{d}{b}. \tag{654}$$

Since we are principally interested in the magnitude of the voltage ratio, we have:

$$\left|\frac{E_2}{E_1}\right| = \frac{MR}{d} \cdot \frac{1}{\left|\frac{\omega_1^2 - \omega^2}{\omega\omega_2} + j\right|} \tag{655}$$

or:

$$\left|\frac{E_2}{E_1}\right| = \frac{MR}{d} \cdot \left|\frac{1}{A + jB}\right| = \frac{MR}{d} \cdot f(\omega). \tag{656}$$

The dependence upon the frequency is contained in the function $f(\omega)$. The constant factor preceding it is in most cases very nearly equal to the ratio of transformation. This is due to the fact that the load resistance is usually large compared to either the primary or secondary transformer resistances, so that:

$$d \cong RL_1. \tag{657}$$

This gives:

$$\frac{MR}{d} = \frac{M}{L_1} \cong \frac{n_2}{n_1}. \tag{658}$$

Before discussing $f(\omega)$, let us consider the real and imaginary portions of the denominator, which we have denoted by A and B respectively. B is a constant and equal to unity. A is a function of ω and of the frequencies ω_1 and ω_2 given by (653) and (654) respectively. These we shall call the **essential frequencies,** because they are usually located so as to include the essential frequency band between them.

In Fig. 96 we show plots of A, B, and $f(\omega)$. The plot of A is rather interesting, and worth noting since it occurs again. It passes through zero at ω_1 and rapidly approaches its asymptote which is a straight line through the origin and the point $(-1,$

ω_2). The frequency at which $A = -1$ is therefore practically ω_2. The frequency at which $A = +1$ is:

$$\frac{\omega_1{}^2}{\omega_2}. \tag{659}$$

At these two frequencies the value of $f(\omega)$ is 0.707. For practical reasons it is essential to have the frequency (659) as low as possible, and ω_2 as high as possible. At the lower essential frequency ω_1, $f(\omega)$ is unity. Above ω_2 it drops off very nearly hyperbolically.

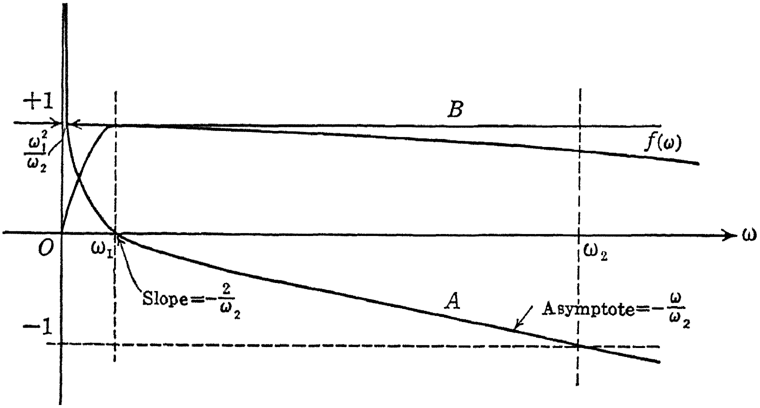


FIG. 96

Whenever (657) is justified, then also:

$$a \cong R_1 R. \tag{660}$$

Then by (653) we see that:

$$\omega_1 = \sqrt{\frac{R_1 R}{L_1 L_2 - M^2}}. \tag{661}$$

For a given load resistance it is advantageous to have the primary resistance R_1 as small as possible.

The ratio of the essential frequencies is an index of the band width. This is:

$$\frac{\omega_1}{\omega_2} = \frac{b\omega_1}{d} = \frac{a}{d\omega_1}$$

or by substituting the approximate values for a and d, we have:

$$\frac{\omega_1}{\omega_2} = \frac{R_1}{L_1\omega_1}. \tag{662}$$

The primary resistance to inductance ratio is therefore an important factor in determining the band width. Incidentally, we note that:

$$\frac{\omega_1^2}{\omega_2} = \frac{R_1}{L_1} \tag{662a}$$

which is the frequency at which $A = +1$ as given by (659). The upper essential frequency is approximately given by:

$$\omega_2 = \frac{RL_1}{L_1L_2 - M^2}. \tag{663}$$

This will be higher for a large load resistance, large primary inductance, and low magnetic leakage. We will close the discussion of this case with a numerical example.

The following data is taken from an iron-core audio transformer. We shall assume here that the saturation in the iron has no effect. We have:

$$\left.\begin{aligned} R_1 &= 1300 \\ R_2 &= 6400 \\ L_1 &= 140 \\ L_2 &= 1400 \\ M &= 414. \end{aligned}\right\} \tag{664}$$

By (662) or (662*a*) we then have:

$$\omega_2 = 0.1077\,\omega_1^2 \tag{665}$$

or:

$$\omega_1 = \sqrt{9.28\,\omega_2}. \tag{665a}$$

Since the two essential frequencies are thus definitely related by the primary transformer constants, the choice of a load resistance will, according to (661) or (663), fix the essential frequency band. We have:

$$\left.\begin{aligned} &R = 91.1\,\omega_2 \\ \text{or:}\quad &R = 9.81\,\omega_1^2 \end{aligned}\right\} \tag{666}$$

so that:

$$\left.\begin{aligned} \omega_1 &= 1.01\sqrt{R} \\ \omega_2 &= 0.011\,R \\ \text{and:}\quad \frac{\omega_2}{\omega_1} &= 0.0109\sqrt{R}. \end{aligned}\right\} \tag{667}$$

For:

$$R = 10^6$$

we get:

$$\left.\begin{aligned} \omega_1 &= 1.01 \cdot 10^3 \\ \omega_2 &= 1.1 \cdot 10^4 \end{aligned}\right\}$$

corresponding to frequencies of:

$$\left.\begin{aligned} f_1 &= 160.5 \\ f_2 &= 1750. \end{aligned}\right\} \tag{668a}$$

On the other hand, for:

$$R = 0.5 \cdot 10^6$$

we get:

$$\left.\begin{aligned} f_1 &= 113.5 \\ f_2 &= 875. \end{aligned}\right\} \tag{669}$$

From these results it is quite apparent that this transformer was either designed to work into a very high resistance load, or into a load of an entirely different character. In order to plot the above results we must determine the factor:

$$\frac{MR}{d}$$

in the expression (656). By (651) we have for the one meg-ohm load:

$$d = 142.7 \cdot 10^6$$

so that:

$$\frac{MR}{d} = \frac{414}{142.7} = 2.9. \tag{670}$$

For the one-half meg-ohm load:

$$d = 72.7 \cdot 10^6$$

so that:

$$\frac{MR}{d} = \frac{414}{145.4} = 2.85. \tag{671}$$

Both plots have the characteristic shape of $f(\omega)$ in Fig. 96 with proper allowance for the correct relative positions of ω_1 and ω_2. Incidentally, the angular frequency ω may be replaced by the actual frequency f in this figure, since the expression for $f(\omega)$ in (655) is dimensionless. The results for this numerical example are shown in Fig. 97. It is very easy to carry out these plots with the help of the characteristic plot of Fig. 96. In fact a rough sketch of them may be made as soon as the essential frequencies ω_1 and ω_2 together with the factor $\frac{MR}{d}$ are known.

We shall next consider a pure capacitive load. For this:

$$Z_L = \frac{1}{jC\omega}. \tag{672}$$

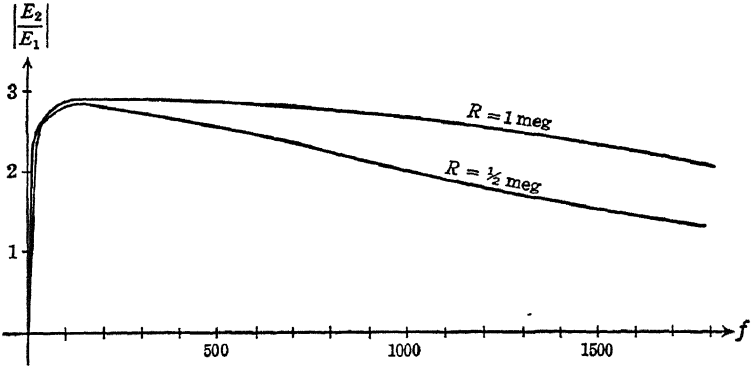


FIG. 97

Substituting this into (649) and arranging terms, we have:

$$\frac{E_2}{E_1} = \frac{-jM\omega}{[R_1 - C(R_1L_2 + R_2L_1)\omega^2] + j\omega[L_1 + CR_1R_2 - C(L_1L_2 - M^2)\omega^2]}. \tag{673}$$

If we let:

$$\left.\begin{aligned} a &= R_1 \\ b &= C(R_1L_2 + R_2L_1) \\ d &= L_1 + CR_1R_2 \\ e &= C(L_1L_2 - M^2). \end{aligned}\right\} \tag{674}$$

Then (673) becomes:

$$\frac{E_2}{E_1} = \frac{-jM\omega}{(a - b\omega^2) + j\omega(d - e\omega^2)}. \tag{675}$$

Comparing this form with (652) for the pure resistance load, we see that the ratio now will have two maxima instead of one. That is, we expect that the voltage ratio will peak at the lower essential frequency:

$$\omega_1^2 = \frac{a}{b} \tag{676}$$

as in the previous case, and again at the upper essential frequency which we now define as:

$$\omega_2^2 = \frac{d}{e}. \tag{677}$$

Thus we expect to get a better overall characteristic with a capacitive load provided the various parameters can be properly adjusted.

In order to obtain a flat characteristic between the two essential frequencies, the corresponding peaks at these frequencies should be equal in height, i.e.:

$$\left|\frac{E_2}{E_1}\right|_{\omega_1} = \left|\frac{E_2}{E_1}\right|_{\omega_2}. \tag{678}$$

By (675) this becomes:

$$\frac{M}{e(\omega_1^2 - \omega_2^2)} = \frac{M\omega_2}{b(\omega_1^2 - \omega_2^2)}$$

or:

$$e\omega_2 = b. \tag{679}$$

Under this condition (675) may be written in terms of the magnitude only:

$$\left|\frac{E_2}{E_1}\right| = \frac{M}{d}\cdot\frac{1}{\left|\dfrac{\omega_1^2 - \omega^2}{\omega\omega_2} + j\dfrac{\omega_2^2 - \omega^2}{\omega_2^2}\right|} \tag{680}$$

or:

$$\left|\frac{E_2}{E_1}\right| = \frac{M}{d}\cdot\frac{1}{|A + jB|} = \frac{M}{d}\cdot f(\omega). \tag{680a}$$

Note the similarity between this form and (655) for the previous case. The factor:

$$\frac{M}{d}$$

is very nearly equal to the transformer ratio because d is usually approximated very closely by:

$$d \cong L_1. \tag{681}$$

The factor A in the denominator of (680a) is the same as for the previous case. The factor B is the new feature about this case, In Fig. 98 we show plots of A, B, and $f(\omega)$. From this we see that $f(\omega)$ does not dip down between the frequencies ω_1 and ω_2, but even rises somewhat above the value unity. Hence, we see that the characteristic is quite flat over the essential band.

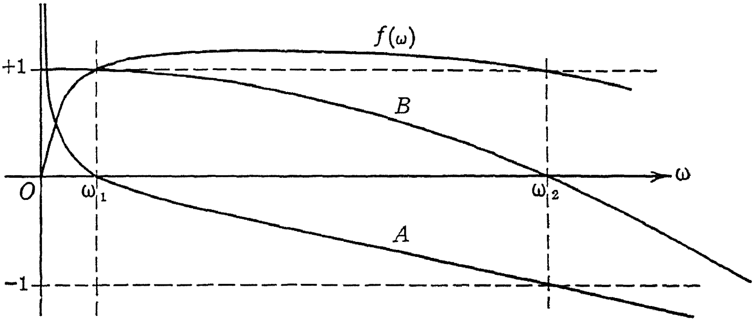


FIG. 98

By (674), (676), and (677) we have for the limits of this band:

$$\omega_1 = \sqrt{\frac{R_1}{C(R_1L_2 + R_2L_1)}} \tag{682}$$

and:

$$\omega_2 = \sqrt{\frac{L_1}{C(L_1L_2 - M^2)}} \tag{683}$$

where the approximate value (681) was used for d. From the last equation we see that ω_2 is that frequency at which resonance occurs between the leakage inductance of the transformer and the capacitance of the load. When the latter is fixed, it is obvious that ω_2 will drop with increasing leakage reactance. Since the leakage reactance is larger in transformers with a high ratio of turns, we see that increasing the latter will lower the upper essential frequency for a fixed capacitive load. This is an important factor which limits the ratio of audio transformers to be used with vacuum tube loads.

Since (679) is equivalent to:

$$d\omega_1^2 = a\omega_2 \tag{679a}$$

we have by using the approximate form for d:

$$\frac{\omega_1}{\omega_2} = \frac{R_1}{L_1\omega_1} \tag{684}$$

which is the same as the relation (662) for a resistance load. Hence we see that the essential frequency range is again limited by the ratio of primary resistance to inductance.

Due to the fact that the three relations (682), (683), and (684) must be simultaneously satisfied in order to give rise to the characteristic ratio shown in Fig. 98, we see that a given transformer must be loaded by a definite capacity in order to obtain a flat voltage curve over the essential frequency range. By combining the above three relations we find:

$$C = \frac{L_1(L_1L_2 - M^2)}{(R_1L_2 + R_2L_1)^2}. \tag{685}$$

This also fixes the essential frequencies, thus:

$$\omega_1 = \sqrt{\frac{R_1(R_1L_2 + R_2L_1)}{L_1(L_1L_2 - M^2)}} \tag{686}$$

and:

$$\omega_2 = \frac{R_1L_2 + R_2L_1}{L_1L_2 - M^2}. \tag{687}$$

For the audio transformer specified by the data (664) we would have:

$$C = 0.242 \cdot 10^{-6} \text{ farads.} \tag{688}$$

Also:

and:

$$\left.\begin{aligned} \omega_1 &= 135.5 \\ \omega_2 &= 213.0. \end{aligned}\right\} \tag{689}$$

From these results it is quite obvious that this particular transformer was not designed to give a flat voltage characteristic when working into a pure capacitive load. The upper frequency ω_2 is much too low. This is due to the fact that the load capacity is too high. If the latter were decreased, then ω_2 would increase as may be seen from (683). But the characteristic would no longer be flat, i.e., it would become much higher at the point ω_2 than at ω_1. This may be seen in the following way.

The relation (678) expresses the condition that the characteristic be flat. If in general we were to write:

$$\left|\frac{E_2}{E_1}\right|_{\omega_2} = r\left|\frac{E_2}{E_1}\right|_{\omega_1} \tag{678a}$$

then it would follow that:

$$e\omega_2 = rb \tag{679b}$$

instead of (679). The result (680a) would be unchanged except for:

$$A = \frac{\omega_1{}^2 - \omega^2}{r\omega\omega_2}. \tag{680b}$$

The expressions (682) and (683) for the essential frequencies remain unchanged. However, the relation (684) now becomes:

$$\frac{\omega_1}{\omega_2} = \frac{rR_1}{L_1\omega_1} \tag{684a}$$

or:

$$r = \frac{L_1\omega_1{}^2}{R_1\omega_2}. \tag{684b}$$

For example, if we terminated the above audio transformer in say:

$$C = 100\ \mu.\mu.\text{f}.$$

then by (682) and (683) we would have:

$$\omega_1 = 2.19 \cdot 10^3$$

and:

$$\omega_2 = 1.05 \cdot 10^4$$

so that by (684b) we would get:

$$r = 49.$$

This means that the ratio of load voltage to primary voltage would be 49 times its normal value at the frequency ω_2. Such a peak can not be tolerated as a general thing. When, however, only a narrow band of frequencies is to be covered, then this peaking property may be made use of in increasing the voltage amplification. The professional vernacular calls this process: "Peaking" a transformer for such and such a frequency band. When a wide band of frequencies is to be covered with uniform results, then the parameters should be so adjusted as to make the value of r as near unity as possible.

Let us next consider a load made up of a resistance and a capacitance in parallel. For this we have:

$$\frac{1}{Z_L} = \frac{1}{R} + jC\omega$$

or:

$$Z_L = \frac{R}{1 + jRC\omega}. \qquad (690)$$

Then the voltage ratio (649) becomes:

$$\frac{E_2}{E_1} = \frac{-jMR\omega}{(a - b\omega^2) + j\omega(d - e\omega^2)} \qquad (691)$$

where we now have:

$$\left.\begin{aligned} a &= R_1(R_2 + R) \\ b &= RC(R_1L_2 + R_2L_1) + L_1L_2 - M^2 \\ d &= RCR_1R_2 + R_1L_2 + R_2L_1 + RL_1 \\ e &= RC(L_1L_2 - M^2). \end{aligned}\right\} \qquad (692)$$

Again we define the essential frequencies:

$$\omega_1^2 = \frac{a}{b} \qquad (693)$$

and:

$$\omega_2^2 = \frac{d}{e}. \qquad (694)$$

If we require that the function have the same magnitude at ω_1 and ω_2, we have as before:

$$e\omega_2 = b \qquad (695)$$

or:

$$d\omega_1^2 = a\omega_2. \qquad (695a)$$

Then the magnitude of (691) becomes:

$$\left|\frac{E_2}{E_1}\right| = \frac{MR}{d} \cdot \frac{1}{\left|\dfrac{\omega_1^2 - \omega^2}{\omega\omega_2} + j\,\dfrac{\omega_2^2 - \omega^2}{\omega_2^2}\right|} \qquad (696)$$

or:

$$\left|\frac{E_2}{E_1}\right| = \frac{MR}{d} \cdot \frac{1}{|A + jB|} = \frac{MR}{d} \cdot f(\omega). \qquad (696a)$$

Here $f(\omega)$ is identical with that for the pure capacitive load, and is, therefore, given by Fig. 98. The essential difference between that case and this lies in the values of the parameters (692). The first three of these may usually be approximated by:

$$\left.\begin{aligned} a &\doteq R_1R \\ b &\cong L_1L_2 - M^2 \\ d &\cong RL_1. \end{aligned}\right\} \tag{692a}$$

In terms of these values we have:

$$\omega_1 = \sqrt{\frac{R_1R}{L_1L_2 - M^2}} \tag{697}$$

and:

$$\omega_2 = \sqrt{\frac{L_1}{C(L_1L_2 - M^2)}}\,. \tag{698}$$

We see that ω_2 is again the frequency at which the leakage inductance resonates with the load capacitance. Consequently the transformer ratio is here also restricted by the upper essential frequency.

The condition (695*a*) for a flat characteristic becomes:

$$\omega_2 = \frac{L_1\omega_1^2}{R_1}$$

or:

$$\frac{\omega_1}{\omega_2} = \frac{R_1}{L_1\omega_1} \tag{699}$$

which is the same as (684) for the capacitive case. If we use (695) to express the condition for a flat characteristic, we have:

$$RC\omega_2 = 1. \tag{700}$$

In this form it links the load resistance and capacitance with the upper essential frequency.

In this case the load parameters are not completely specified by the transformer constants as in the case of a pure capacitive load. This is due to the fact that we have the *three* equations (697), (698), and either (699) or (700) which interrelate the *four* quantities:

$$\omega_1 \qquad \omega_2 \qquad R \qquad C.$$

Hence, any one of these may be specified arbitrarily. In order to illustrate this, let us take the audio transformer constants (664) again. We will arbitrarily specify that the lower essential frequency shall be determined by:

$$\omega_1 = 500. \tag{701}$$

Then we have from (699) that:

$$\omega_2 = 27000. \tag{702}$$

Using these values in (697) and (698), we find that:

and:

$$\left.\begin{aligned} R &= 2.5 \cdot 10^6 \\ C &= 15 \cdot 10^{-12}. \end{aligned}\right\} \tag{703}$$

The voltage ratio characteristic for this transformer with a load consisting of the above resistance and capacitance in parallel is given in Fig. 99. This figure is very easily drawn by modifying $f(\omega)$ in Fig. 98 to comply with the values for the essential frequencies given by (701) and (702), and multiplying the ordinates by the factor:

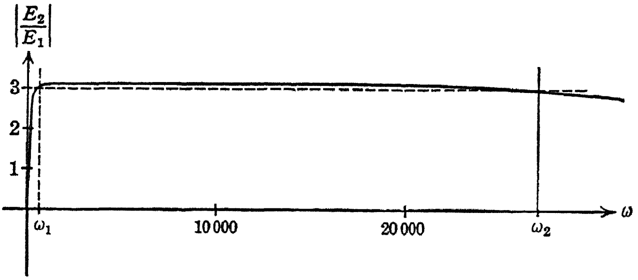


FIG. 99

$$\frac{MR}{d} \cong \frac{M}{L_1} = 2.96. \tag{704}$$

Instead of arbitrarily fixing ω_1 we could have chosen a value for any of the other three quantities involved. From the results of this case it would seem advisable to arbitrarily put a resistance in parallel with a pure capacitive load in order to improve the voltage characteristic of the transformer.

The charactéristic may be made peaked in this case also. For the voltage ratio at ω_2 to be r times what it is at ω_1 we have the same relations (697) and (698) as before, but instead of (699) or (700) we find:

$$\frac{\omega_1}{\omega_2} = \frac{rR_1}{L_1\omega_1} \tag{699a}$$

and:

$$RC\omega_2 = r. \tag{700a}$$

In this case the factor A in (696*a*) becomes:

$$A = \frac{\omega_1^2 - \omega^2}{r\omega\omega_2} \tag{696b}$$

just as in the previous capacitive case. Now, out of the five quantities:

$$\omega_1 \qquad \omega_2 \qquad R \qquad C \qquad r$$

any two may be arbitrarily fixed.

Finally, we shall treat one more type of load which is quite common, namely that of a resistance and capacitance in series. For this we have:

$$Z_L = \frac{1 + jRC\omega}{jC\omega}. \tag{705}$$

Substitution into (649) gives:

$$\frac{E_2}{E_1} = \frac{-jM\omega(1 + jRC\omega)}{(a - b\omega^2) + j\omega(d - e\omega^2)} \tag{706}$$

with:

$$\left.\begin{aligned} a &= R_1 \\ b &= C(R_1L_2 + R_2L_1 + RL_1) \\ d &= L_1 + CR_1(R_2 + R) \\ e &= C(L_1L_2 - M^2). \end{aligned}\right\} \tag{707}$$

The essential frequencies are again given by:

$$\omega_1^2 = \frac{a}{b} \tag{708}$$

and:

$$\omega_2^2 = \frac{d}{e}. \tag{709}$$

Then (706) may be written:

$$\frac{E_2}{E_1} = \frac{-jM}{d} \cdot \frac{1 + jRC\omega}{\frac{b}{d}\frac{\omega_1^2 - \omega^2}{\omega} + j\frac{\omega_2^2 - \omega^2}{\omega_2^2}}. \tag{710}$$

The magnitude of the voltage ratio at the two essential frequencies, therefore, becomes:

$$\left|\frac{E_2}{E_1}\right|_{\omega_1} = \frac{M}{d} \frac{|1 + jRC\omega_1|}{\dfrac{\omega_2^2 - \omega_1^2}{\omega_2^2}} \tag{711}$$

and:

$$\left|\frac{E_2}{E_1}\right|_{\omega_2} = \frac{M}{d} \frac{|1 + jRC\omega_2|}{\dfrac{\omega_2^2 - \omega_1^2}{\omega_2} \cdot \dfrac{b}{d}} \cdot \tag{712}$$

Now we write the condition that the ratio at ω_2 should be r times what it is at ω_1, i.e.:

$$\left|\frac{E_2}{E_1}\right|_{\omega_2} = r\left|\frac{E_2}{E_1}\right|_{\omega_1} \cdot \tag{713}$$

This gives:

$$|1 + jRC\omega_2|d = |1 + jRC\omega_1|rb\omega_2 \tag{713a}$$

which is equivalent to:

$$R^2C^2\omega_2^2 = -\frac{1 - r^2\omega_2^2\left(\dfrac{b}{d}\right)^2}{1 - r^2\omega_1^2\left(\dfrac{b}{d}\right)^2} \cdot \tag{714}$$

This result is not particularly useful in the above form because b and d contain the quantities R and C. However, the relative magnitudes of these quantities and the transformer constants are usually such that the following approximations are permissible:

$$\left.\begin{array}{l} b \cong RCL_1 \\ d \cong L_1. \end{array}\right\} \tag{715}$$

Then (714) may be solved for the RC product explicitly in terms of ω_1, ω_2, and r. We get:

$$R^2C^2 = \frac{-\omega_2(r^2 - 1) + \sqrt{\omega_2^2(r^2 - 1)^2 + 4\, r^2\omega_1^2}}{2\, r^2\omega_1^2\omega_2} \cdot \tag{716}$$

We are particularly interested in the result for $r = 1$. This gives:

$$RC\sqrt{\omega_1\omega_2} = 1. \tag{717}$$

It is interesting to compare this with the result (700) which expresses the same condition for the case of a resistance and capacitance in parallel. The general condition for a peaked characteristic at ω_2, given by (716), is now much more complicated than (700a), which expresses this same condition for the parallel load.

With the approximations (715) we get for the essential frequencies (708) and (709):

$$\omega_1 = \sqrt{\frac{R_1}{RCL_1}} \tag{718}$$

and:

$$\omega_2 = \sqrt{\frac{L_1}{C(L_1L_2 - M^2)}}\,. \tag{719}$$

Note that ω_2 is still the resonant frequency for the leakage inductance and the load capacitance.

When a flat characteristic is desired for this case, we consider the equations (717), (718), and (719). Out of the four quantities:

$$\omega_1 \qquad \omega_2 \qquad R \qquad C$$

any one may be arbitrarily specified. This assumes, of course, that a transformer with definite constants is given.

On the other hand, when a peaked characteristic is desired, we consider (716), (718), and (719). Then the five quantities:

$$\omega_1 \qquad \omega_2 \qquad R \qquad C \qquad r$$

are involved, of which any two may be arbitrarily specified.

Note, incidentally, that by combining (717) and (718) we get:

$$\frac{\omega_1}{\omega_2} = \left(\frac{R_1}{L_1\omega_1}\right)^2. \tag{720}$$

Comparing this with (699) for the parallel resistance and capacitance load, we see that the essential frequency band is now wider for the same ratio of primary resistance to inductance.

We still have to put the expression (710) for the voltage ratio into its final form. If we define:

$$\sqrt{\omega_1\omega_2} = \omega_m \tag{721}$$

as the geometric mean frequency, then we have for the magnitude of the voltage ratio:

$$\left|\frac{E_2}{E_1}\right| = \frac{M}{L_1} \cdot \frac{\left|1 + j\frac{\omega}{\omega_m}\right|}{\left|\frac{\omega_1{}^2 - \omega^2}{\omega\omega_m} + j\frac{\omega_2{}^2 - \omega^2}{\omega_2{}^2}\right|} \tag{722}$$

or:

$$\left|\frac{E_2}{E_1}\right| = \frac{M}{L_1} \cdot \left|\frac{P + jQ}{A + jB}\right| = \frac{M}{L_1} \cdot f(\omega). \tag{722a}$$

This is somewhat different than (696) for the parallel load. The factor B is the only one which is not changed. Fig. 100 illustrates the quantities P, Q, A, B, and $f(\omega)$. We see that the latter is quite flat and somewhat greater than unity over the frequency range between ω_1 and ω_2. The values at ω_1 and ω_2 are equal, as they should be since the expression (722) was derived on this basis.

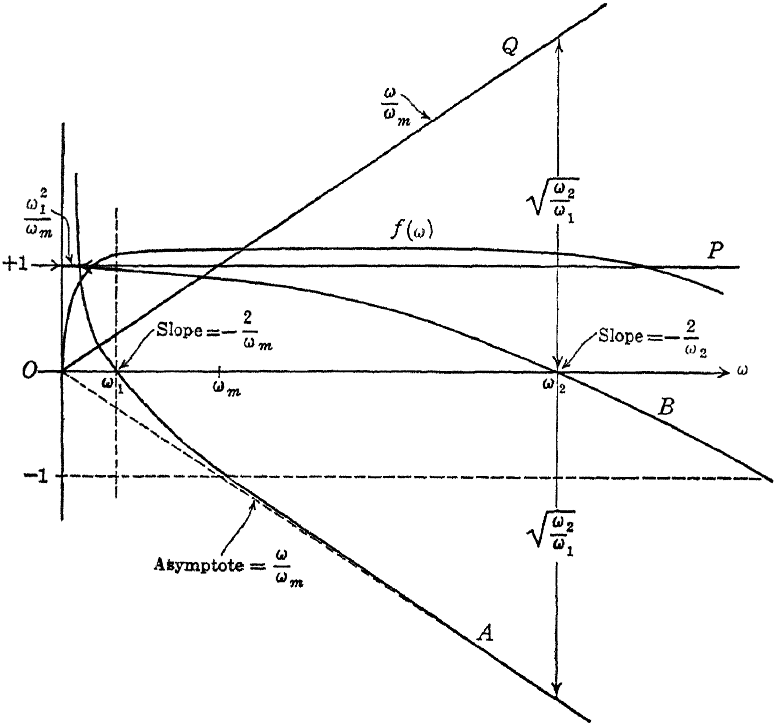


FIG. 100

It is interesting to note in connection with this case that the relation (716) reduces to:

$$RC\omega_2 = 1 \tag{723}$$

for:

$$r = \sqrt{2}. \tag{724}$$

The condition (723) for the parallel resistance and capacitance load led to a flat characteristic. Here it leads to a characteristic which is 1.414 times as high at ω_2 as it is at ω_1.

We will close the discussion of this case with a numerical example, using the transformer data given by (664) again. Let us determine that load for which the voltage ratio characteristic will be flat. Since we may choose one of the quantities ω_1, ω_2, R, C arbitrarily, we will fix ω_1 at:

$$\omega_1 = 150. \tag{725}$$

From (720) we obtain:

$$\omega_2 = 1.16 \cdot 10^{-2}\omega_1^3 \tag{726}$$

so that we find:

$$\omega_2 = 39100. \tag{727}$$

Thus by (718) and (719) we have:

and:

$$\left.\begin{aligned} R &= 5.76 \cdot 10^7 \\ C &= 7.17 \cdot 10^{-12}. \end{aligned}\right\} \tag{728}$$

Fig. 101 shows a plot of the voltage characteristic with this load.

We could continue to take up other types of loads and work out the conditions for a flat characteristic, but the above will suffice to give the reader an idea of the essential features of the problem.

Whenever a capacitive load is involved, the upper essential frequency is limited by the leakage inductance of the transformer referred to the secondary side. This is:

$$\frac{L_1L_2 - M^2}{L_1} = L_2\left(1 - \frac{M^2}{L_1L_2}\right). \tag{729}$$

The factor:

$$\left(1 - \frac{M^2}{L_1L_2}\right) = \sigma \tag{730}$$

is often called the **leakage coefficient** of the transformer. Its value is usually in the neighborhood of from five to seven per cent for iron core audio transformers. In terms of this factor we have, for example, by (719):

$$C = \frac{1}{\sigma L_2 \omega_2{}^2} \tag{731}$$

and by (718):

$$R = \frac{\sigma L_2 \omega_2{}^2}{\omega_m}. \tag{732}$$

Practically, we wish to have ω_2 as high as possible. In order to keep C from getting too small and R from getting too large, the leakage coefficient must be small. It is usually not small enough. The other alternative would be to make L_2 small, but with fixed L_1 this means a decreased ratio of transformation. Increasing the latter, increases L_2, so that the ratio is limited by ω_2 also. Decreasing the inductances of the transformer tends to make the exciting current comparable to, or perhaps larger than the load current. This tends to make the current ratio appreciably different from the inverse ratio of turns, as pointed out earlier in connection with eq. (635), so that the impedance looking into the primary terminals is no longer the transformed load impedance, but substantially the impedance of the primary winding. This may be seen from the network of Fig. 94 as represented by the parameters (629). The input impedance at the primary terminals is given by:

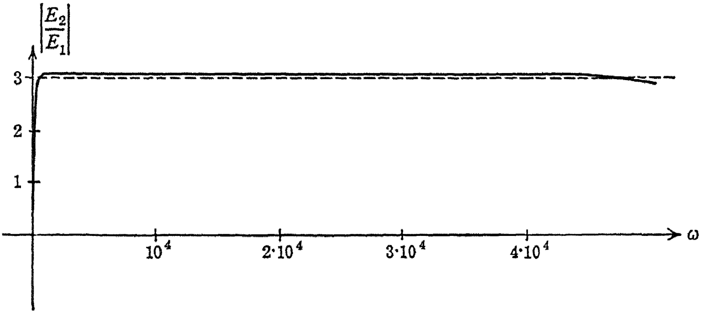


FIG. 101

$$Z_{11} = \frac{D}{B_{11}} \tag{733}$$

where D is the network determinant, and B_{11} the minor of its first row and column. Thus:

$$Z_{11} = \frac{(R_1 + jL_1\omega)(R_2 + jL_2\omega + Z_L) + M^2\omega^2}{R_2 + jL_2\omega + Z_L}. \tag{733a}$$

If the inductive reactances, as well as the resistances, are small compared to the load impedance Z_L over the essential frequency range, then we have approximately:

$$Z_{11} \cong \frac{(R_1 + jL_1\omega)Z_L}{Z_L} = R_1 + jL_1\omega \tag{734}$$

as stated above. On the other hand, when the inductive reactances predominate, we have:

$$Z_{11} \cong \frac{(M^2 - L_1L_2)\omega^2 + jL_1\omega Z_L}{jL_2\omega} \tag{735}$$

and since the leakage is small:

$$Z_{11} \cong \frac{L_1}{L_2} \cdot Z_L \cong \left(\frac{n_1}{n_2}\right)^2 \cdot Z_L. \tag{735a}$$

The condition which gives rise to (734) is usually undesirable because it leads to a very small input impedance which is substantially inductive. If it were not for this objection, then it would be advisable to keep the transformer inductances low, because a better voltage ratio characteristic results.

In conclusion we wish to point out that the above analysis of the transformer network omits one factor which is of considerable importance practically, and that is the effect of distributed capacity in the windings. We treated the network as a lumped-constant problem, which means that we took into account only those constants which are predominant at relatively low frequencies. At high frequencies, such as are encountered in the vicinity of the upper end of the audio band, the distributed capacity between turns and layers of the windings begins to become effective in influencing the behavior of the transformer. It is, however, very difficult to take this effect into account analytically. Attempts have been made in this direction by replacing the distributed capacity by a lumped capacity in parallel with the transformer winding. Such a procedure is usually inaccurate for two reasons:

First, no reliable data is available for the value of this equivalent lumped capacitance; and second, its use is justified only over a limited range of frequencies which is hard to specify. The best we can do, therefore, is to base our analysis upon the lumped-constant assumption, and then modify the results thus obtained by what we can gather from experimental familiarity with the problem.

2. Transformer Equivalent Circuits. We wish now to take up a subject which sometimes aids in getting a clearer understanding of the properties of a transformer. This consists essentially of replacing the given network involving the transformer, by another network of new physical build but identical characteristics. In presenting this subject we are broaching a new branch of network theory altogether, namely that of equivalent networks. This subject is one of very great interest as well as of unlimited utility. It is, incidentally, a subject which has just recently been more fully understood. Briefly, it centers about the fact that, whereas a given network possesses a unique set of characteristics, the converse is not true. That is, when a suitable set of characteristics is specified, it is possible to find an infinite number of networks which will possess this set of characteristics. If this is so, then for any given network there exist an infinite set of other networks which will have the same characteristics. It may very well be that some of these equivalent networks are preferable for reasons of analysis or practice, and therefore methods for determining them are worth developing.

In a later chapter we shall go into this interesting subject very thoroughly, because it has great possibilities in the field of design work. Here we shall restrict our discussion to a few simple cases which suffice to represent the transformer with its associated load. Our starting point is the Fig. 95 which we have made use of earlier. With the same notation regarding the transformer constants, we now define:

$$\left.\begin{aligned} b_{11} &= R_1 + jL_1\omega \\ b_{22} &= R_2 + jL_2\omega \\ b_{12} &= jM\omega = b_{21}. \end{aligned}\right\} \qquad (736)$$

The voltage E_2 is an impressed voltage for the load, and hence acts in the direction of the secondary current. It is given by

(640) in terms of the load current and impedance. We shall consider it in this sense here. If we arbitrarily define the mutual inductance as negative numerically, then we may write for the equilibrium conditions:

$$\left.\begin{aligned} b_{11}I_1 - b_{12}I_2 &= E_1 \\ -b_{21}I_1 + b_{22}I_2 &= E_2. \end{aligned}\right\} \tag{737}$$

The reason for thus defining M and placing minus signs before the b_{12} terms is perfectly arbitrary, and is done for convenience only. This will be seen shortly.

We wish to show that one type of network, which can be made equivalent to the transformer, is given by Fig. 102. It consists essentially of the so-called T-network, characterized by the impedance arms Z_A, Z_B, and Z_C. The ideal transformer is added to this structure for the sake of generality. This will be appreciated later. Namely, we shall see that various special forms result by giving the ratio a of the ideal transformer particular values.

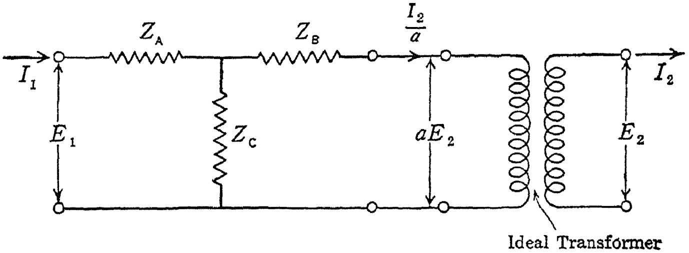


Fig. 102

The principal feature of the equivalent network is that the T-structure contains no purely mutual inductances. It seems to be a failing of our minds to prefer networks without purely mutual inductances. At least for purposes of visualization, we seem to have trouble in seeing how any current gets from the primary of a transformer to the secondary without any visible means of support. Mathematically, this trouble does not exist; and for analytic reasons we should hardly require a T-structure to replace the transformer of Fig. 95. It is helpful, however, in clarifying some of the physical concepts pointed out earlier in this chapter, and sometimes allows the transformer to be actually replaced by another circuit when used in conjunction with the proper load.

For the T-structure alone we may write the equilibrium conditions:

$$\left.\begin{aligned} (Z_A + Z_C)I_1 - \frac{Z_C}{a}I_2 &= E_1 \\ -Z_CI_1 + \frac{Z_B + Z_C}{a}I_2 &= aE_2 \end{aligned}\right\}$$

or:

$$\left.\begin{aligned} (Z_A + Z_C)I_1 - \frac{Z_C}{a}I_2 &= E_1 \\ -\frac{Z_C}{a}I_1 + \frac{Z_B + Z_C}{a^2}I_2 &= E_2. \end{aligned}\right\} \qquad (738)$$

If this network together with the ideal transformer is to be identical in behavior to the transformer of Fig. 95, then the systems (737) and (738) must be identical. This obviously requires that:

$$\left.\begin{aligned} \frac{Z_C}{a} &= b_{12} \\ Z_A + Z_C &= b_{11} \\ \frac{Z_B + Z_C}{a^2} &= b_{22} \end{aligned}\right\} \qquad (739)$$

which gives:

$$\left.\begin{aligned} Z_A &= b_{11} - ab_{12} \\ Z_B &= a^2b_{22} - ab_{12} \\ Z_C &= ab_{12}. \end{aligned}\right\} \qquad (740)$$

By assigning any value we please to the ratio a, we immediately have the impedances in the T-structure to be used in conjunction with an ideal transformer having this ratio, in order to replace the given transformer. Thus, we see that an infinite number of equivalent circuits exist. We shall discuss several of the more common forms.

Case I. The simplest form results when we make $a = 1$, i.e., the ideal transformer vanishes. Then we have by (740):

$$\left.\begin{aligned} Z_A &= b_{11} - b_{12} \\ Z_B &= b_{22} - b_{12} \\ Z_C &= b_{12} \end{aligned}\right\} \qquad (741)$$

and by (736) this is:

$$\left.\begin{aligned} Z_A &= R_1 + j\omega(L_1 - M) \\ Z_B &= R_2 + j\omega(L_2 - M) \\ Z_C &= j\omega M. \end{aligned}\right\} \tag{741a}$$

Fig. 103 illustrates this result. The shunt branch of the T contains only the mutual inductance. The series branches consist of the resistances and the differences between the self- and mutual inductances. It should be noted that unless the ratio of the original transformer is unity, one of these differences will be negative. When:

$$\left.\begin{aligned} &n_2 > n_1 \\ \text{then:}\quad &L_2 > M > L_1 \end{aligned}\right\} \tag{742}$$

unless the leakage flux is excessive. On the other hand, when:

$$\left.\begin{aligned} &n_1 > n_2 \\ \text{then:}\quad &L_1 > M > L_2. \end{aligned}\right\} \tag{743}$$

The fact that one of the inductances in the equivalent T-network comes out negative makes it impossible to realize this circuit by means of ordinary physical elements. However, so long as the above representation is used merely for analytic work, this objection is not serious, since a negative inductance is readily created on paper by means of a minus sign. There are cases where this circuit may be used actually, as well as analytically. Suppose, for example, that $L_2 - M$ came out negative, but that the transformer was to work into a load having an inductance of L_L henries. Then so long as:

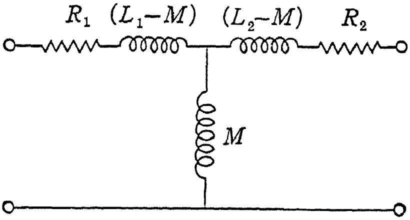


FIG. 103

$$L_L \geqq M - L_2 \tag{744}$$

the negative inductance in the T-circuit can be combined with the positive inductance of the load to form a resulting inductance which is either zero or positive. Thus, by modifying the load, it would be physically possible to replace the transformer by the above equivalent circuit.

From this discussion it is apparent that this particular equivalent circuit is generally dissymmetrical, i.e., that in general:

$$Z_A \neq Z_B.$$

If we demand that the equivalent circuit be symmetrical, then we have:

Case II, for which we put

$$Z_A = Z_B. \tag{745}$$

By (740) we then have:

$$a = \sqrt{\frac{b_{11}}{b_{22}}} = \sqrt{\frac{R_1 + jL_1\omega}{R_2 + jL_2\omega}}. \tag{746}$$

If the resistances are small compared to the inductive reactances over the essential frequency band, then the ratio of the ideal transformer becomes approximately:

$$a \cong \sqrt{\frac{L_1}{L_2}} \cong \frac{n_1}{n_2}. \tag{746a}$$

Substituting (746) back into (740), we get:

$$\left.\begin{aligned} Z_A &= b_{11} - b_{12}\sqrt{\frac{b_{11}}{b_{22}}} \\ Z_B &= b_{11} - b_{12}\sqrt{\frac{b_{11}}{b_{22}}} \\ Z_C &= b_{12}\sqrt{\frac{b_{11}}{b_{22}}} \end{aligned}\right\} \tag{747}$$

or:

$$\left.\begin{aligned} Z_A = Z_B &= R_1 + jL_1\omega - jM\omega\sqrt{\frac{R_1 + jL_1\omega}{R_2 + jL_2\omega}} \\ Z_C &= jM\omega\sqrt{\frac{R_1 + jL_1\omega}{R_2 + jL_2\omega}}. \end{aligned}\right\} \tag{747a}$$

The original transformer with a load impedance equal to Z_L is then replaced by the network of Fig. 104, where the parameters are given by (746) and (747*a*). This case is interesting analytically, but not physically realizable unless the approximation used in (746*a*) is allowable. Then (747*a*) becomes:

$$\left.\begin{aligned} Z_A = Z_B &\cong R_1 + j\omega L_1\left(1 - \frac{M}{\sqrt{L_1L_2}}\right) \\ Z_C &\cong j\omega L_1 \cdot \frac{M}{\sqrt{L_1L_2}}. \end{aligned}\right\} \qquad (748)$$

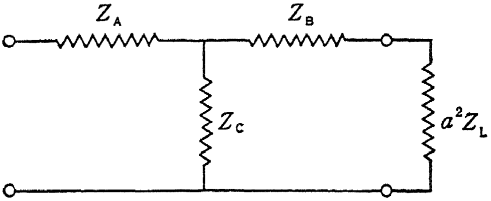


FIG. 104

But by (730) we see that:

$$\left.\begin{aligned} \sigma = \left(1 - \frac{M^2}{L_1L_2}\right) &= \left(1 - \frac{M}{\sqrt{L_1L_2}}\right)\left(1 + \frac{M}{\sqrt{L_1L_2}}\right) \\ &\cong 2\left(1 - \frac{M}{\sqrt{L_1L_2}}\right) \end{aligned}\right\} \qquad (730a)$$

so that (748) becomes:

$$\left.\begin{aligned} Z_A = Z_B &\cong R_1 + j\omega L_1 \cdot \frac{\sigma}{2} \\ Z_C &\cong j\omega L_1\left(1 - \frac{\sigma}{2}\right). \end{aligned}\right\} \qquad (748a)$$

Hence, we see that the total series inductance in the T-circuit of Fig. 104 is substantially equal to the primary inductance times the

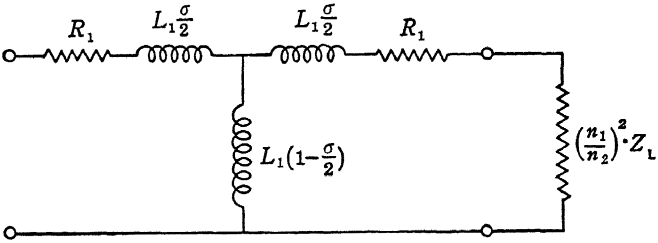

R_1
$L_1\frac{\sigma}{2}$
$L_1\frac{\sigma}{2}$
R_1
$L_1(1-\frac{\sigma}{2})$
$\left(\frac{n_1}{n_2}\right)^2 \cdot Z_L$

FIG. 105

leakage coefficient, i.e., the total leakage inductance referred to the primary side. The total series resistance is twice the primary resistance, i.e., the primary resistance plus the secondary resistance referred to the primary. The shunt impedance of the T-circuit is practically equal to the primary inductive reactance, and the load impedance is also referred to the primary. Thus, it appears that this equivalent network refers everything to the primary side. The approximate circuit may be realized physically as illustrated in Fig. 105. For purposes of approximate analysis this equivalent circuit may be very useful. It may also be useful for making actual tests in the laboratory provided the modified load can be constructed.

Case III. We shall now consider the simplified equivalent circuit that results when we demand that:

$$Z_A = 0. \tag{749}$$

By the first equation (740) this gives:

$$a = \frac{b_{11}}{b_{12}} \tag{750}$$

from which we see that:

and:

$$\left.\begin{aligned} Z_B &= b_{11}\left(\frac{b_{11}b_{22}}{b_{12}^2} - 1\right) \\ Z_C &= b_{11}. \end{aligned}\right\} \tag{751}$$

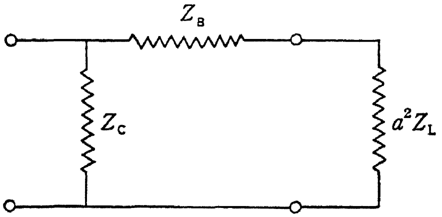


FIG. 106

This equivalent circuit is shown in Fig. 106, where the constants are given by (750) and (751) above. This type of circuit is often called a half-section or an L-section.

This circuit may also be realized physically when the resistances are negligible compared to the inductive reactances over the essential band. Then (750) becomes:

$$a \cong \frac{L_1}{M} \cong \frac{n_1}{n_2}. \tag{750a}$$

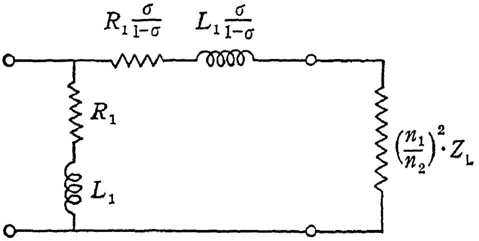
For the first equation (751) we find:

$$Z_B = (R_1 + jL_1\omega) \cdot \frac{\sigma}{1 - \sigma}. \tag{751a}$$

Hence, the approximate equivalent circuit becomes that shown in

Fig. 107. The load is again referred to the primary side. The total series inductance is also substantially equal to the leakage inductance referred to the primary side, because:

$$\frac{\sigma}{1-\sigma} \cong \sigma.$$



$R_1 \frac{\sigma}{1-\sigma}$
$L_1 \frac{\sigma}{1-\sigma}$
R_1
L_1
$\left(\frac{n_1}{n_2}\right)^2 \cdot Z_L$

FIG. 107

Note, incidentally, that the impedance looking into the primary with the secondary open is:

$$R_1 + jL_1\omega$$

in all the Figs. 103, 105, and 107 for the various cases treated so far. Since this must be so, it is clear that the shunt impedance in this last case must contain the resistance R_1.

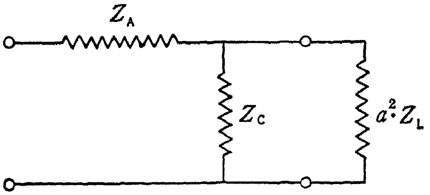


FIG. 108

Case IV. Finally, we shall demand that:

$$Z_B = 0. \tag{752}$$

By the second equation (740) this gives:

$$a = \frac{b_{12}}{b_{22}} \tag{753}$$

so that we have:

$$\left.\begin{aligned} Z_A &= b_{11}\left(1 - \frac{b_{12}^2}{b_{11}b_{22}}\right) \\ Z_C &= \frac{b_{12}^2}{b_{22}}. \end{aligned}\right\} \tag{754}$$

This equivalent circuit is illustrated in Fig. 108, where the parameters are given by (753) and (754).

Here again we may derive an approximate equivalent circuit by neglecting the resistances in comparison with the inductive reactances over the essential band. We then have by (753):

$$a \cong \frac{M}{L_2} \cong \frac{n_1}{n_2} \tag{753a}$$

and by (754):

and:

$$\left.\begin{aligned} Z_A &= (R_1 + jL_1\omega)\sigma \\ Z_C &= (R_1 + jL_1\omega)\,(1 - \sigma). \end{aligned}\right\} \tag{754a}$$

This situation is illustrated in Fig. 109. If the leakage were negligible, we would simply have the transformed-load impedance

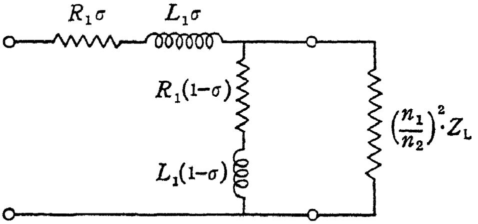


FIG. 109

in parallel with the primary winding impedance. From this it is obvious that the impedance looking into the primary will be substantially equal to the transformed-load impedance only when

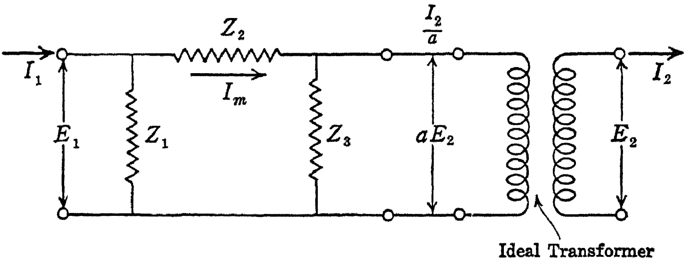


FIG. 110

the latter is small compared to the primary winding impedance, as pointed out earlier.

Another type of equivalent circuit which is considered more or less fundamental, is the so-called Π-network. This, together with its ideal transformer, is shown in Fig. 110. The Π-circuit is inherently more complicated than the T-circuit, because it constitutes essentially a three-mesh network when associated with its simple load. It is a little more involved analytically to treat this case because of the introduction of the current I_m. However, this can easily be eliminated when we consider that the voltage around the central mesh must vanish, i.e., that:

$$-Z_1I_1 + (Z_1 + Z_2 + Z_3)I_m - Z_3\frac{I_2}{a} = 0. \tag{755}$$

It is also easy to see that:

$$\left.\begin{aligned}(I_1 - I_m)Z_1 &= E_1\\ \left(I_m - \frac{I_2}{a}\right)Z_3 &= -aE_2.\end{aligned}\right\} \tag{756}$$

From (755) and (756) we get:

$$\left.\begin{aligned}\frac{Z_1(Z_2 + Z_3)}{\sum Z}I_1 - \frac{Z_1Z_3}{a\sum Z}I_2 &= E_1\\ -\frac{Z_1Z_3}{a\sum Z}I_1 + \frac{Z_3(Z_1 + Z_2)}{a^2\sum Z} &= E_2\end{aligned}\right\} \tag{757}$$

where:

$$\sum Z = Z_1 + Z_2 + Z_3. \tag{758}$$

The system (757) must be equivalent to (737). This requires that:

$$\left.\begin{aligned}\frac{Z_1(Z_2 + Z_3)}{\sum Z} &= b_{11}\\ \frac{Z_1Z_3}{a\sum Z} &= b_{12}\\ \frac{Z_3(Z_1 + Z_2)}{a^2\sum Z} &= b_{22}\end{aligned}\right\} \tag{759}$$

which is equivalent to:

$$\left.\begin{aligned}\frac{Z_1Z_2}{\sum Z} &= b_{11} - ab_{12}\\ \frac{Z_2Z_3}{\sum Z} &= a^2b_{22} - ab_{12}\\ \frac{Z_3Z_1}{\sum Z} &= ab_{12}\end{aligned}\right\} \tag{760}$$

or:

$$\left.\begin{aligned} Z_1 &= \frac{a(b_{11}b_{22} - b_{12}^2)}{ab_{22} - b_{12}} \\ Z_2 &= \frac{a(b_{11}b_{22} - b_{12}^2)}{b_{12}} \\ Z_3 &= \frac{a^2(b_{11}b_{22} - b_{12}^2)}{b_{11} - ab_{12}} \cdot \end{aligned}\right\} \tag{761}$$

By giving a in (761) any value, we obtain the impedances to be used in the Π-structure in Fig. 110 which, together with the ideal transformer of ratio a, will replace the original transformer of Fig. 95.

It should be noted, incidentally, that by comparing (760) with (740) we see that:

$$\left.\begin{aligned} Z_A &= \frac{Z_1Z_2}{\sum Z} \\ Z_B &= \frac{Z_2Z_3}{\sum Z} \\ Z_C &= \frac{Z_3Z_1}{\sum Z} \end{aligned}\right\} \tag{762}$$

which are the well-known relations between the T and Π impedances for equivalent networks.[1]

We shall now discuss the results (761) for several special cases. First we have:

Case I, for which we put:

$$a = 1. \tag{763}$$

The corresponding impedances are apparent from (761). The structure is not symmetrical because:

$$Z_1 \neq Z_3.$$

We may construct an approximate equivalent for this case when, over the essential frequency range, it is permissible to put:

$$1 - \frac{b_{12}^2}{b_{11}b_{22}} \cong 1 - \frac{M^2}{L_1L_2} = \sigma. \tag{764}$$

---

[1] Kennelly, A. E., "The Equivalence of Triangles and Three-pointed Stars in Conducting Networks," *Elec. World and Eng.*, Vol. XXXIV, p. 413, 1899.

Then we have by (761), (763), and (764):

$$\left.\begin{aligned} \frac{1}{Z_1} &\cong \frac{1}{b_{11}\sigma} - \frac{1}{b_{12}\sigma} \\ Z_2 &\cong b_{12}\sigma \\ \frac{1}{Z_3} &\cong \frac{1}{b_{22}\sigma} - \frac{1}{b_{12}\sigma} \end{aligned}\right\} \tag{765}$$

where terms above the first power in σ have been dropped.

Fig. 111 illustrates this result. The impedances Z_1 and Z_3 are parallel combinations as indicated by the first and third equations

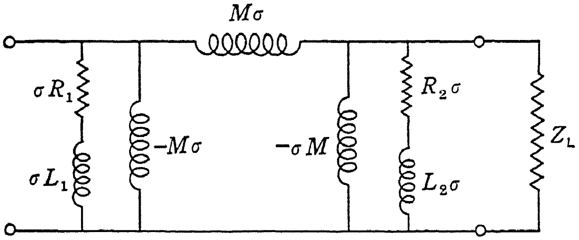


FIG. 111

(765). The network is not physically realizable by simple means on account of the negative inductance:

$$-M\sigma.$$

Case II. Here we demand that the structure be symmetrical, i.e., that:

$$Z_1 = Z_3. \tag{766}$$

By (761) this gives:

$$a = \sqrt{\frac{b_{11}}{b_{22}}} \tag{767}$$

and also:

$$\left.\begin{aligned} Z_1 = Z_3 &= b_{11}\left(1 + \frac{b_{12}}{\sqrt{b_{11}b_{22}}}\right) \\ Z_2 &= b_{11}\left(\frac{\sqrt{b_{11}b_{22}}}{b_{12}} - \frac{b_{12}}{\sqrt{b_{11}b_{22}}}\right). \end{aligned}\right\} \tag{768}$$

This form is merely of theoretical interest. However, when the approximation (764) is allowable, then we may also put:

$$a \cong \sqrt{\frac{L_1}{L_2}} \cong \frac{n_1}{n_2} \tag{767a}$$

and (768) becomes:

$$\left.\begin{aligned} Z_1 = Z_3 &= (R_1 + jL_1\omega)(2 - \sigma) \\ Z_2 &= (R_1 + jL_1\omega)\sigma. \end{aligned}\right\} \tag{768a}$$

This result is illustrated in Fig. 112.

The cases for $Z_3 \to \infty$ and $Z_1 \to \infty$ are identical with Cases III and IV above, and will, therefore, not be worked out here. There are, of course, many other values which we might choose for a in order to obtain equivalent circuits which may be useful in special cases. We shall let the reader continue the above ideas as a source of practice problems.

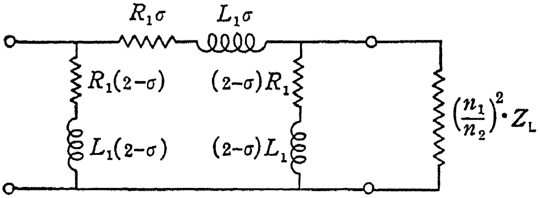


FIG. 112

So far we have treated only the transformer with pure inductive coupling. The auto-transformer may be treated in exactly the same manner. With the notation indicated in Fig. 113 we have:

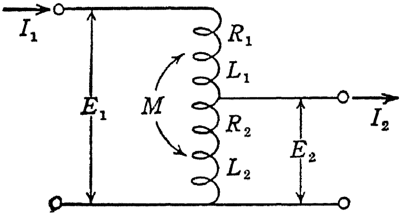


FIG. 113

$$\left.\begin{aligned} b_{11} &= R_1 + R_2 + j\omega(L_1 + L_2 + 2M) \\ b_{22} &= R_2 + jL_2\omega \\ b_{12} &= R_2 + j\omega(L_2 + M) = b_{21}. \end{aligned}\right\} \tag{769}$$

The above formulae for the ordinary transformer may be applied to the auto-transformer of Fig. 113 with the values (769) substituted in place of those given by (736). There is not sufficient novelty about this process to warrant its introduction here.

3. The Steady-State Characteristics of Tuned-Coupled Circuits. We come now to another problem of fundamental importance in communication work. This concerns itself with the behavior of a series of coupled oscillatory circuits which are individually tuned to the same frequency. At the close of Chapter III we treated the resonance condition in a single R, L, C circuit. We saw there that this circuit exhibited selective properties with regard to the frequencies impressed upon it. We shall now treat the selective properties of several inductively coupled tuned circuits. This is essentially a steady-state investigation. Later we shall take up the natural behavior of tuned-coupled circuits.

Let us consider first the case of two meshes inductively coupled, as illustrated by Fig. 114. Both meshes contain the same values of R, L, and C. The inductive coupling is indicated by M. We assume that the relative magnitudes of the parameters are such that the network is fully oscillatory. Then it will possess two pairs of conjugate modes, each of which defines one natural frequency and one decrement. For the steady state we are not primarily interested in the latter. The natural frequencies, however, are essential to our problem, because the selective properties which we are interested in, depend upon the resonance phenomena of the network. These in turn depend upon the natural frequencies. Whenever an impressed frequency coincides with one of the latter, the response of the network becomes exceptional. In order to have the network be selective with respect to a desired frequency band, it is necessary to so select the parameters as to group the natural frequencies in the vicinity of this band. How this may be most effectively done will now be taken up.

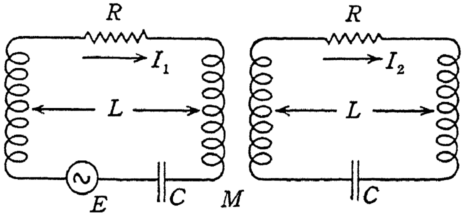


FIG. 114

The mesh and mutual impedances of our two-mesh network become:

$$\left.\begin{aligned} b_{11} &= b_{22} = R + j\left(L\omega - \frac{1}{C\omega}\right) \\ b_{12} &= b_{21} = jM\omega \end{aligned}\right\} \qquad (770)$$

where ω is the impressed angular frequency. The time-average equilibrium is expressed by:

$$\left.\begin{aligned} b_{11}I_1 + b_{12}I_2 &= E \\ b_{21}I_1 + b_{22}I_2 &= 0. \end{aligned}\right\} \tag{771}$$

The current in mesh #2 is the one we are interested in because this will be the load current in any practical case. For this we have:

$$I_2 = \frac{B_{12}E}{D} \tag{772}$$

where we see from the relations (770) above that:

$$D = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}^2 - b_{12}^2$$

or:

$$D = (b_{11} + b_{12})\,(b_{11} - b_{12}) \tag{773}$$

and:

$$B_{12} = -b_{12}. \tag{774}$$

Substituting from (770) into (773), we get:

$$D = \left\{R + j\left[(L+M)\omega - \frac{1}{C\omega}\right]\right\}\left\{R + j\left[(L-M)\omega - \frac{1}{C\omega}\right]\right\}. \tag{775}$$

The first of these factors is resonant at:

$$\omega_1 = \frac{1}{\sqrt{C(L+M)}} \tag{776}$$

and the second at:

$$\omega_2 = \frac{1}{\sqrt{C(L-M)}}. \tag{777}$$

Since the product of these two resonant factors is the denominator of the expression (772), we see that we have to expect two resonance maxima for the current I_2 at approximately the frequencies indicated by (776) and (777), respectively. The maxima will in general not lie exactly at those frequencies because the numerator of (772) is also a function of frequency, and hence will affect the resultant current somewhat. However, this latter effect is usually very slight and may be neglected in the following discussion.

If we write for (775):

$$D = (R + jX_1)\,(R + jX_2) \tag{775a}$$

then the reactances X_1 and X_2 become the vital parameters. In Fig. 115 these are shown plotted against ω. We see that X_1 becomes asymptotic to the straight line $(L + M)\omega$, whereas X_2 becomes asymptotic to $(L - M)\omega$. The frequency indicated by ω_0 is that frequency for which the sum of the two curves becomes zero, i.e., for which:

$$X_1 + X_2 = 0. \tag{778}$$

This gives:

$$2\left(L\omega - \frac{1}{C\omega}\right) = 0$$

or:

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{779}$$

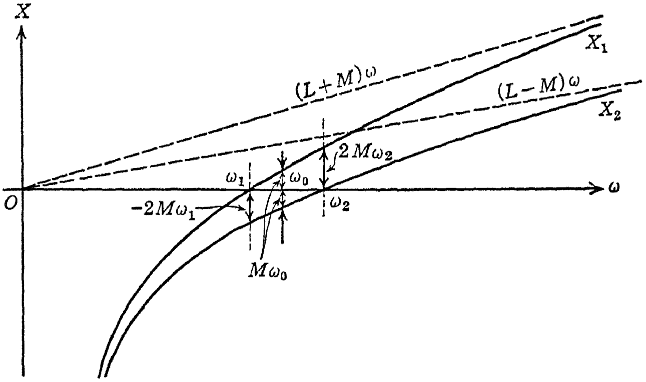


FIG. 115

which is the resonant frequency of a single R, L, C circuit. This frequency is very nearly the geometric mean between ω_1 and ω_2, which is:

$$\sqrt{\omega_1\omega_2} = \frac{1}{\sqrt[4]{C^2(L^2 - M^2)}} \tag{780}$$

or if we expand this in a Taylor series, we have:

$$\sqrt{\omega_1\omega_2} = \frac{1}{\sqrt{LC}}\left\{1 - \frac{1}{4}\left(\frac{M}{L}\right)^2 + \cdot \cdot \cdot\right\}. \tag{780a}$$

Usually terms above the first power in:

$$\frac{M}{L}$$

may be neglected, so that:

$$\sqrt{\omega_1\omega_2} \cong \omega_0. \tag{780b}$$

The expression (772) for the current in the second mesh, therefore becomes:

$$I_2 = \frac{-jM\omega E}{(R + jX_1)\,(R + jX_2)}\cdot \tag{781}$$

In order to plot the current against frequency, it is convenient to derive the expressions for the values at the frequencies ω_1, ω_2, and ω_0. We have by (775), (775*a*), (776), and (777):

$$X_2(\omega_1) = \frac{L - M}{\omega_1}(\omega_1^2 - \omega_2^2) \tag{782}$$

and:

$$X_1(\omega_2) = \frac{L + M}{\omega_2}(\omega_2^2 - \omega_1^2). \tag{782a}$$

Also:

$$X_1(\omega_0) = -X_2(\omega_0) = M\omega_0. \tag{783}$$

If we care only for the magnitude of the current, then we have by (781), (782), and (783):

$$|I_2|_{\omega_1} = \frac{E}{R}\cdot\frac{M\omega_1^2}{\sqrt{R^2\omega_1^2 + (L - M)^2(\omega_2^2 - \omega_1^2)^2}}\cdot \tag{784}$$

Similarly, we have:

$$|I_2|_{\omega_2} = \frac{E}{R}\cdot\frac{M\omega_2^2}{\sqrt{R^2\omega_2^2 + (L + M)^2(\omega_2^2 - \omega_1^2)^2}}\cdot \tag{785}$$

Finally, at ω_0 we get:

$$|I_2|_{\omega_0} = \frac{M\omega_0 E}{R^2 + M^2\omega_0^2}\cdot \tag{786}$$

Usually (784) and (785) are maxima, i.e., the resonance maxima, and (786) is the intervening minimum. When the mutual inductance is not exceptionally small, the resistance term under the radical in the expressions (784) and (785) is negligible, so that:

$$|I_2|_{\omega_1} \cong \frac{E}{R}\cdot\frac{M}{L - M}\cdot\frac{\omega_1^2}{\omega_2^2 - \omega_1^2} \tag{784a}$$

and:

$$|I_2|_{\omega_2} \cong \frac{E}{R}\cdot\frac{M}{L + M}\cdot\frac{\omega_2^2}{\omega_2^2 - \omega_1^2}\cdot \tag{785a}$$

Then also:

$$|I_2|_{\omega_0} \cong \frac{E}{M\omega_0}. \tag{786a}$$

In terms of these forms, a numerical case may be easily carried through. For example, suppose we have:

$$\left.\begin{aligned} L &= 10^{-3} \\ C &= 10^{-9} \\ R &= 5 \\ M &= 10^{-4}. \end{aligned}\right\} \tag{787}$$

Then:

$$L - M = 0.9 \cdot 10^{-3} \qquad L + M = 1.1 \cdot 10^{-3}$$

$$\omega_1 = \frac{10^6}{\sqrt{1.1}} = 0.953 \cdot 10^6$$

$$\omega_2 = \frac{10^6}{\sqrt{0.9}} = 1.053 \cdot 10^6$$

$$\omega_1^2 = 0.905 \cdot 10^{12}$$

$$\omega_2^2 = 1.11 \cdot 10^{12}$$

$$\omega_2^2 - \omega_1^2 = 0.205 \cdot 10^{12}$$

$$\omega_0 = 10^6 \qquad \omega_0^2 = 10^{12}.$$

Hence, by (784*a*), (785*a*), and (786*a*) we have:

$$|I_2|_{\omega_1} = \frac{E}{5} \cdot \frac{10^{-4}}{0.9 \cdot 10^{-3}} \cdot \frac{0.905}{0.205} = 0.0982\,E$$

$$|I_2|_{\omega_2} = \frac{E}{5} \cdot \frac{10^{-4}}{1.1 \cdot 10^{-3}} \cdot \frac{1.11}{0.205} = 0.0983\,E$$

$$|I_2|_{\omega_0} = 0.01\,E.$$

The complete plot of $|I_2|$ vs. ω is given in Fig. 116.

In general we can see from the above analysis that the two resonance frequencies ω_1 and ω_2 will come closer together with decreasing mutual inductance. Also from the expression (786*a*) we see that the minimum between the resonance maxima will increase with decreasing M. However, so far as being able to visualize the complete behavior regarding relative magnitudes of the parameters is concerned, the above solution is not particularly satisfying. It seems to be fairly compact, and yet it is hard to see by means of it how the various maxima and minima change with changing M and R, for example. The latter two are the vital parameters, by means of which the selectivity of the network is controlled. We must obviously put the solution to this case in a more compact and convenient form in order to facilitate a thorough investigation of these matters. This we can do by applying the following approximate method.

Let us return again to the expression (775) for the determinant of our system. Consider, for example, the factor:

$$\left[(L + M)\omega - \frac{1}{C\omega}\right]. \tag{788}$$

We are interested in how this factor changes with ω only in the vicinity of the frequency ω_0. In tuned radio circuits, for example, ω_0 is very high, and the width of the band which we wish to select is very narrow compared to this frequency. Also, the mutual inductance is usually very small compared to the self-inductance. With this in mind we write:

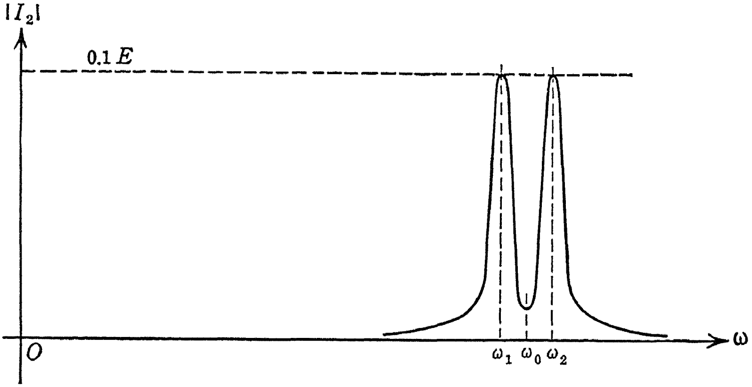


Fig. 116

$$\omega = \omega_0 + \delta \tag{789}$$

where δ is a small increment in frequency which may be either positive or negative. Now we recognize that:

$$\begin{aligned}\frac{1}{C(\omega_0 + \delta)} &\cong \frac{1}{C\omega_0}\left(1 - \frac{\delta}{\omega_0}\right)\\ &\cong \frac{1}{C\omega_0} - L\delta\end{aligned} \tag{790}$$

where higher powers of δ are neglected, and the relation:

$$\frac{1}{\omega_0^2} = LC$$

is made use of. Then (788) becomes:

$$L\omega_0 + M\omega_0 + L\delta + M\delta - \frac{1}{C\omega_0} + L\delta. \tag{791}$$

Here the term $M\delta$ is a small quantity of the second order and hence negligible, so that we have:

$$X_1 = 2\,L\delta + M\omega_0. \tag{792}$$

Now we introduce the change of variable:

$$\delta = \frac{R}{2\,L}x = \alpha x \tag{793}$$

where we recognize α as the decrement of the simple R, L, C circuit. Next we introduce the notation:

$$y = \frac{M\omega_0}{R}\cdot \tag{794}$$

These things are done merely for convenience. The expression (792) then reads:

$$X_1 = R(x + y). \tag{795}$$

The meaning of x is clear from a comparison of (789) and (793). Thus, we see that $x = 0$ is the frequency ω_0. Increasing or decreasing x in units, moves our point of interest up or down the frequency scale in units of α measured from ω_0 as an origin.

In exactly the same way we get:

$$X_2 = R(x - y) \tag{796}$$

because X_2 is exactly the same as X_1 except for the negative mutual, which changes the sign of y. Hence, the determinant (775) becomes:

$$D = R^2\,[1 + j(x + y)]\,[1 + j(x - y)]. \tag{797}$$

The magnitude of D becomes:

$$|D| = R^2\,\sqrt{\{1 + (x + y)^2\}\,\{1 + (x - y)^2\}}. \tag{797a}$$

So far as the minor B_{12} is concerned we have:

$$|B_{12}| = |b_{12}| = M\omega$$

and in the vicinity of ω_0 we neglect its dependence upon ω altogether and get by (794):

$$|B_{12}| = M\omega_0 = Ry. \tag{798}$$

This is consistent with the above, because $M\delta$ is of the second order. Thus, the current according to (772) becomes:

$$|I_2| = \frac{yE}{R\sqrt{\{1 + (x + y)^2\}\{1 + (x - y)^2\}}}\cdot \tag{799}$$

This expression is very convenient. From it we may easily study the selectivity as a function of the damping and the mutual inductance. The units in which x varies the frequency about ω_0 are equal to the decrement; and the parameter y is a direct measure of the mutual inductance. It also has an interesting significance in terms of the resonant frequencies given by (776) and (777). If we expand the radicals in these expressions, we see that:

$$\left.\begin{aligned} \omega_1 &\cong \omega_0\left(1 - \frac{M}{2L}\right) \\ \text{and:}\qquad \omega_2 &\cong \omega_0\left(1 + \frac{M}{2L}\right)\cdot \end{aligned}\right\} \tag{800}$$

By (794) we then have:

$$\left.\begin{aligned} \omega_1 &\cong \omega_0 - \alpha y \\ \text{and:}\qquad \omega_2 &\cong \omega_0 + \alpha y. \end{aligned}\right\} \tag{801}$$

From this we see that the resonant frequencies ω_1 and ω_2 are $\pm\, y$ units removed from ω_0 respectively. This means that:

$$\left.\begin{aligned} \omega_1 \text{ corresponds to } x &= -y \\ \omega_2 \text{ corresponds to } x &= +y. \end{aligned}\right\} \tag{802}$$

If in particular we made $y = 1$, then the resonant frequencies would be one unit each removed from the mean frequency ω_0. Thus, the band width may easily be studied. This state of affairs is illustrated by Fig. 116a, where the frequency band is laid out symmetrically about ω_0 in terms of x and y. The maxima at ω_1 and ω_2 are here labelled I_{-y} and I_y, respectively, while the value at ω_0 is denoted by I_0.

Since the denominator of the expression (799) is symmetrical with respect to x and y, the values I_y and I_{-y} are equal. These may easily be studied as a function of the mutual inductance by letting $x = \pm y$ in (799). We then have:

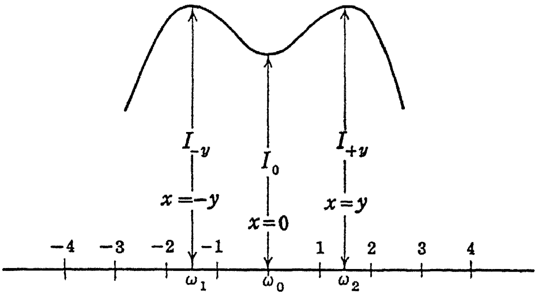


FIG. 116*a*

$$I_{\pm y} = \frac{yE}{R\sqrt{1+4y^2}}. \tag{803}$$

Similarly the value I_0 may be studied by putting $x = 0$, thus:

$$I_0 = \frac{yE}{R(1+y^2)}. \tag{804}$$

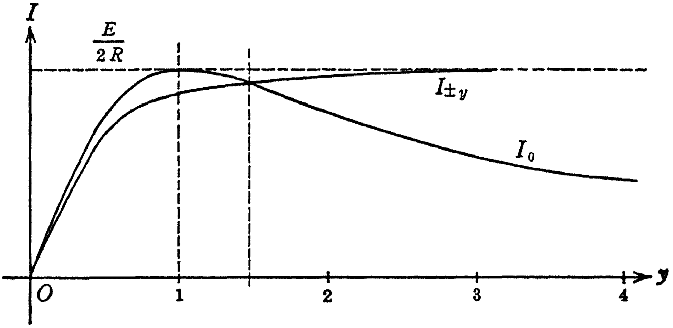


FIG. 117

If we plot these expressions against y, then we have a complete survey of how these important amplitudes vary with the coupling. This is shown in Fig. 117. Here we see that I_0 rises rapidly with

y, reaches the value $\frac{E}{2R}$ at $y = 1$, and then drops to zero asymptotically. $I_{\pm y}$, on the other hand, starts out with the same slope as I_0 and rises to the value $\frac{E}{2R}$ asymptotically. From this plot we see a very interesting thing; namely, that for certain values of coupling I_0 is greater than $I_{\pm y}$. For our numerical problem above this was certainly not the case, because M was too large for this condition. Another interesting point is brought out by the fact that the curves cross each other. At this point the amplitudes I_0 and $I_{\pm y}$ are equal. This gives rise to an almost flat-topped characteristic which is very desirable for selective circuits designed for band tuning. The condition which gives rise to this situation is given by equating (803) and (804). From this we get:

$$y = \sqrt{2} \tag{805}$$

and by (794) this is:

$$M = \frac{\sqrt{2}\,R}{\omega_0} = R\sqrt{2\,LC}. \tag{806}$$

The frequency increment between ω_1 and ω_2 for this condition is $2\sqrt{2}$ units or:

$$\omega_2 - \omega_1 = \sqrt{2}\frac{R}{L}. \tag{807}$$

For the numerical illustration above we have:

$$y = \frac{10^2}{5} = 20.$$

A glance at Fig. 117 shows that this gives rise to $I_{\pm y}$ values almost equal to the asymptote, and an I_0 nearly equal to zero, which checks with what we found previously. In Fig. 118 we have plotted three curves for the same numerical data except with:

$$\left.\begin{array}{ll} y = 1.414 & M = 7.07 \cdot 10^{-6} \\ y = 1.0 & M = 5 \cdot 10^{-6} \\ y = 0.707 & M = 3.54 \cdot 10^{-6} \end{array}\right\} \tag{808}$$

respectively. These plots are very readily carried out by means of the forms (799), (803), and (804).

We see that the one for $y = \sqrt{2}$ is not so flat as it might be. The amplitudes at $x = \pm\sqrt{2}$ and $x = 0$ are equal, but the humps at ± 1 prevent the top from being flat. The curve for $y = 1$ looks more favorable, and also has somewhat straighter sides. The curve for $y = 0.707$ is narrower, but the maximum is beginning to drop. As the mutual is decreased still more, the maximum drops rapidly as shown by Fig. 117.

An interesting point in connection with the above is the fact that the maxima do not coincide with the points $x = \pm y$; i.e., they do not occur exactly at the frequencies for which the reactances X_1 and X_2 vanish, as assumed in the derivation of the relations (776) and (777) for these frequencies. The curve for $y = \sqrt{2}$, for example, does not have maxima at $x = \pm\sqrt{2}$ but at $x = \pm 1$ instead. We may clear up this point very nicely by asking ourselves: For what combinations of x and y do maxima occur? This question is answered either by differentiating the expression (799) with respect to x and equating to zero or by determining the relation between x and y for which:

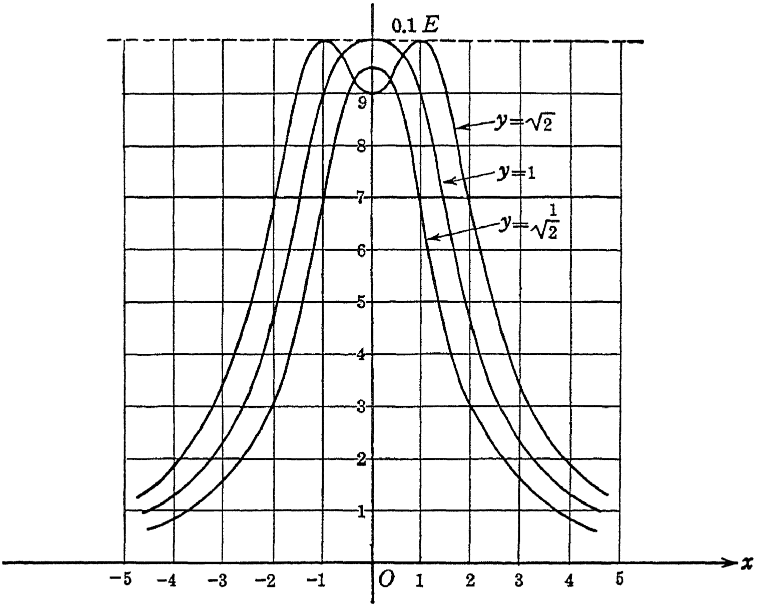


FIG. 118

$$|I_2| = \frac{E}{2R}. \tag{809}$$

By substituting from (799) into the above we get:

$$y^2 = x^2 + 1 \tag{810}$$

as the desired relation. This is a rectangular hyperbola with $y = \pm x$ as asymptotes, and is illustrated in Fig. 119. For any value of y, we can pick off those two values of x for which the resonance curve becomes a maximum, i.e., touches the line:

$$\frac{E}{2R}.$$

We see that $y = 1$ gives $x = 0$; and $y = 1.414$ gives $x = \pm 1$, as borne out by the Fig. 118. Note that no solution exists for $y < 1$, which means that for a mutual inductance less than that corresponding to $y = 1$, the curve does not rise to the maximum line at all. $y = 1$ is the smallest mutual for which the curve will reach the maximum value. It is also interesting to note that for larger values of y, the curve of Fig. 119 comes very close to its asymptote so that practically:

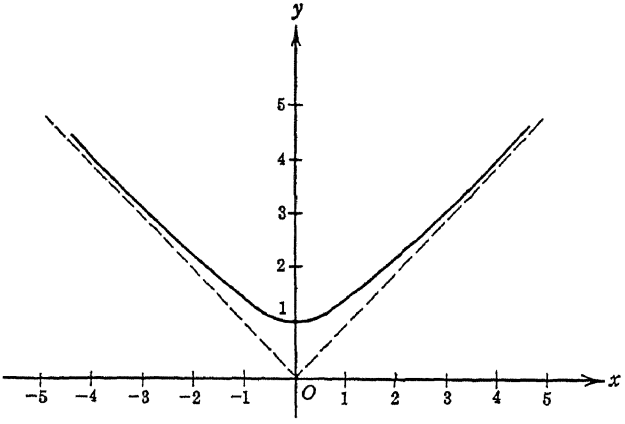


FIG. 119

$$y = x.$$

This means that for larger values of mutual inductance, the maxima practically coincide with the points $x = \pm y$ as originally assumed. The deviation from this becomes noticeable only for values of y in the neighborhood of unity.

From the above analysis it should be a simple matter to determine the optimum value of M for any given case. The same general idea may be carried out for more coupled meshes. It should be noted in comparing the above with the resonance conditions in a single mesh, as worked out in Chapter III, that the curves are sharper for two meshes than for one, i.e., they have straighter sides. This is due to the fact that the expression (799) drops off as:

$$\frac{1}{x^2}$$

for large values of x, whereas the corresponding expression for a single mesh drops off only as:

$$\frac{1}{x}$$

for large values of x. Hence, two meshes are more selective than one. Also the top may be made broader by properly adjusting the value of M. Incidentally, the curves will drop off more sharply for small values of y than for larger values. Consequently, this should be kept in mind also in adjusting M.

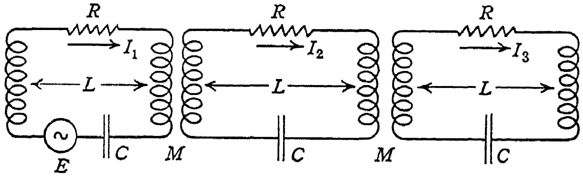


FIG. 120

We continue with the case of three inductively coupled tuned circuits, as illustrated in Fig. 120. Here we have for the mesh and mutual impedances:

$$\left.\begin{aligned} b_{11} &= b_{22} = b_{33} = R + j\left(L\omega - \frac{1}{C\omega}\right) \\ b_{12} &= b_{23} = b_{21} = b_{32} = jM\omega. \end{aligned}\right\} \tag{811}$$

The time-average equilibrium conditions are:

$$\left.\begin{aligned} b_{11}I_1 + b_{12}I_2 + 0 \quad &= E \\ b_{21}I_1 + b_{22}I_2 + b_{23}I_3 &= 0 \\ 0 + b_{32}I_2 + b_{33}I_3 \quad &= 0. \end{aligned}\right\} \tag{812}$$

The current in mesh #3 becomes:

$$I_3 = \frac{B_{13}E}{D} \tag{813}$$

with:

$$D = \begin{vmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{vmatrix} = b_{11}(b_{11}^2 - 2\,b_{12}^2) \tag{814}$$

and:

$$B_{13} = b_{12}^2. \tag{815}$$

The relation (814) may be further modified so that we have:

$$D = b_{11}(b_{11} + \sqrt{2}\,b_{12})\,(b_{11} - \sqrt{2}\,b_{12}). \tag{814a}$$

The three factors in this last expression are respectively:

$$\left.\begin{array}{l} R + j\left(L\omega - \dfrac{1}{C\omega}\right) \\ R + j\left[(L + \sqrt{2}\,M)\omega - \dfrac{1}{C\omega}\right] \\ R + j\left[(L - \sqrt{2}\,M)\omega - \dfrac{1}{C\omega}\right]. \end{array}\right\} \tag{816}$$

If we let:

$$\omega = \omega_0 + \frac{R}{2\,L}x \tag{817}$$

and:

$$y = \frac{\sqrt{2}\,M\omega_0}{R} \tag{818}$$

with:

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{819}$$

then, by means of an expansion process exactly like the one for the two-mesh case above, we find for these three factors:

$$\left.\begin{array}{l} R(1 + jx) \\ R[1 + j(x + y)] \\ R[1 + j(x - y)]. \end{array}\right\} \tag{816a}$$

The magnitude of the determinant is, therefore, given by:

$$|D| = R^3\sqrt{(1 + x^2)[1 + (x + y)^2]\,[1 + (x - y)^2]}. \tag{820}$$

For the magnitude of B_{13} we have:

$$|B_{13}| = M^2\omega_0^2 = \frac{R^2y^2}{2}\cdot \tag{821}$$

Hence the magnitude of the current in mesh #3 becomes:

$$|I_3| = \frac{y^2E}{2\,R\,\sqrt{(1+x^2)[1+(x+y)^2]\,[1+(x-y)^2]}}\cdot \tag{822}$$

Here we expect three maxima because the network possesses three natural frequencies. By the form of (822), we expect these to lie approximately at the points:

$$\left.\begin{aligned} x &= 0 \\ x &= +y \\ x &= -y. \end{aligned}\right\} \tag{823}$$

For these we get:

$$I_0 = \frac{y^2E}{2\,R(1+y^2)} \tag{824}$$

and:

$$I_{\pm y} = \frac{y^2E}{2\,R\,\sqrt{(1+y^2)\,(1+4\,y^2)}}\cdot \tag{825}$$

Plots of these are shown in Fig. 121, from which we see that the asymptote of I_0 is twice that for $I_{\pm y}$. This means that for increasing y, the maximum for $x = 0$ becomes twice as large as the maxima for $x = \pm y$.

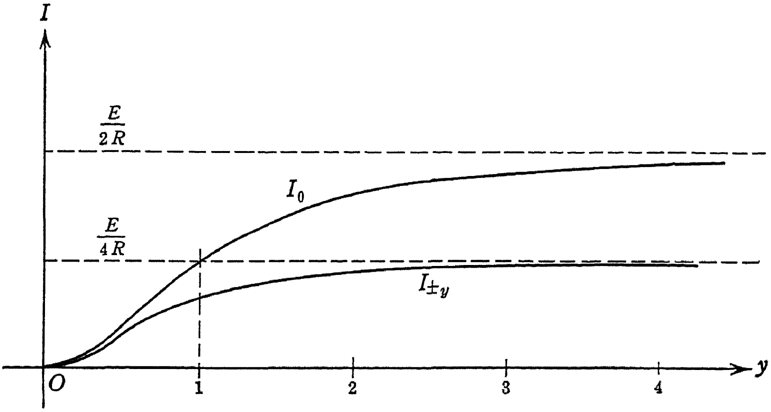


FIG. 121

In order to obtain the relation between x and y for which maxima occur, we differentiate the expression (822) and equate to zero. Thus, we obtain the following:

$$\left.\begin{aligned} x &= 0 \\ x^2 &= \frac{2\,y^2 - 3 \pm y\,\sqrt{y^2 - 12}}{3}\,. \end{aligned}\right\} \tag{826}$$

One maximum always occurs at $x = 0$ regardless of the coupling, and is given by (824). The others are located symmetrically about the point $x = 0$, but do not occur at all unless:

$$y^2 > 12. \tag{827}$$

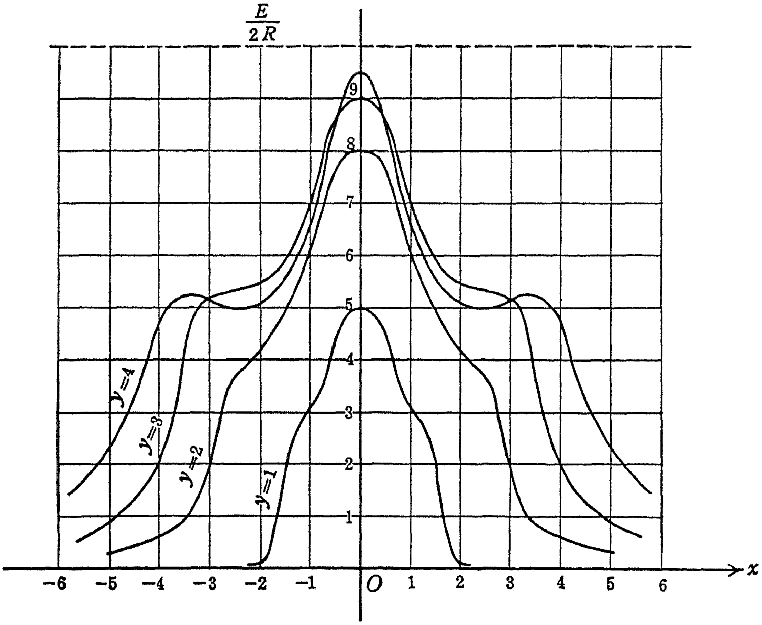


FIG. 122

For large values of y, the second relation (826) yields:

$$x = \pm y \tag{828}$$

and:

$$x = \pm \frac{y}{\sqrt{3}}\,. \tag{829}$$

The first of these is due to the plus sign in (826), and corresponds to the maxima of the current. The second is due to the minus sign, and locates the minima between the central and the two outside maxima. (828) agrees with the approximate relations (823).

For very loose coupling the latter become incorrect, and (826) must be used.

In Fig. 122 are plotted four curves for this case with the numerical data given by(787) but with:

$$\left.\begin{array}{ll} y = 1 & M = 3.54 \cdot 10^{-6} \\ y = 2 & M = 7.07 \cdot 10^{-6} \\ y = 3 & M = 10.61 \cdot 10^{-6} \\ y = 4 & M = 14.14 \cdot 10^{-6} \end{array}\right\} \qquad (830)$$

respectively. Compared with the two-mesh case, these curves look rather peculiar. They do not indicate that the selective properties of a three-mesh chain are particularly good. They are too wide at the base and too pointed at the top for good band selection. Incidentally, the condition (827) for the existence of the outside maxima is well brought out.

Chains of tuned circuits consisting of more than three meshes may be analyzed in a similar manner to the above. We shall not go into this any further, but will now discuss some of the transient properties of tuned-coupled circuits. For this purpose we must investigate the natural behavior of such networks in terms of their normal modes and normal coördinates.

4. The Natural Behavior of Tuned-Coupled Circuits. In the following discussion of various cases, we shall consider the sudden application of a constant force to the chain of tuned circuits instead of the application of a harmonic force. In the next chapter we shall show that the behavior of a network in consequence of the sudden application of a constant force, is characteristically indicative of the behavior under the application of a force of arbitrary form. In fact it is more instructive to study the behavior of a network under the influence of a suddenly-applied constant force than it is to study its behavior when a force of special form is suddenly applied. Another reason for studying the present problem with a constant force rather than a harmonic one is on account of the analytic simplification which results. So far as the normal modes and the relative amplitudes of their corresponding oscillations, as well as the normal coördinates and their geometric interpretation, are concerned, these are independent of the nature of the disturbing force anyway. These points are of chief interest to us, and will now be studied in detail.

For the two-mesh network of Fig. 114 we have the system of condition equations:

$$\left.\begin{aligned} c_{11}J_1 + c_{12}J_2 &= 0 \\ c_{21}J_1 + c_{22}J_2 &= 0 \end{aligned}\right\} \tag{831}$$

where:

$$\left.\begin{aligned} c_{11} = c_{22} &= Lp + R + \frac{1}{Cp} \\ c_{12} = c_{21} &= Mp. \end{aligned}\right\} \tag{832}$$

The modular determinant is therefore:

$$D(p) = (c_{11}c_{22} - c_{12}c_{21}) = (c_{11} + c_{12})\,(c_{11} - c_{12}). \tag{833}$$

The determinantal equation becomes:

$$p^2(c_{11} + c_{12})\,(c_{11} - c_{12}) = 0 \tag{834}$$

which is factorable into:

$$p(c_{11} + c_{12}) = 0 \tag{834a}$$

and:

$$p(c_{11} - c_{12}) = 0. \tag{834b}$$

Each of these is a quadratic equation in p. For the fully oscillatory case, which we shall treat, the modes will consist of two pairs of conjugates, each pair defining one decrement, one natural frequency, and one normal coördinate. Let us denote the pair of modes defined by (834*a*) by:

$$\left.\begin{aligned} p &= p_1 \\ p &= p_2 \end{aligned}\right\} \tag{835}$$

and those corresponding to (834*b*) by:

$$\left.\begin{aligned} p &= p_3 \\ p &= p_4. \end{aligned}\right\} \tag{836}$$

The pair (835) defines the normal coördinate $N_{1,2}$; and the pair (836) defines the normal coördinate $N_{3,4}$. In order to determine the geometric position of either one of these normal coördinates, we may make use of the modular vector group defined by either one of the pair of modes corresponding to that coördinate. That is, we may, for example, use either p_1 or p_2 to determine the position of $N_{1,2}$. Let us use p_1 for $N_{1,2}$, and p_3 for $N_{3,4}$.

Since by (834*a*) for p_1 we have:

$$c_{11}^{(1)} = -c_{12}^{(1)} \tag{837}$$

the modular vector group for p_1 becomes:

$$\left.\begin{array}{lcc} c_1^{(1)} \text{ with components:} & -c_{12}^{(1)} & c_{12}^{(1)} \\ c_2^{(1)} \text{ with components:} & c_{12}^{(1)} & -c_{12}^{(1)}. \end{array}\right\} \tag{838}$$

This pair of vectors is illustrated in Fig. 123. We see that they are equal and opposite, and in the 45° positions in the second and fourth quadrants. Hence, the normal coördinate $N_{1,2}$ must be in the 45° position in the first and third quadrants.

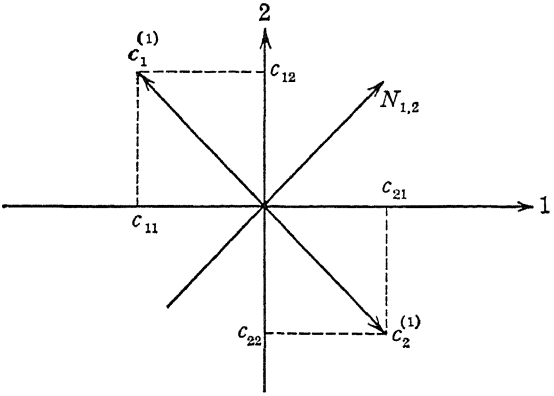


FIG. 123

For the mode p_3 we have by (834*b*):

$$c_{11}^{(3)} = c_{12}^{(3)} \tag{839}$$

so that the corresponding vector group is given by:

$$\left.\begin{array}{lcc} c_1^{(3)} \text{ with components:} & c_{12}^{(3)} & c_{12}^{(3)} \\ c_2^{(3)} \text{ with components:} & c_{12}^{(3)} & c_{12}^{(3)}. \end{array}\right\} \tag{840}$$

This group is illustrated in Fig. 124. Both vectors lie in the 45° position in the first quadrant. This puts the normal coördinate $N_{3,4}$ in the 45° position in the second and fourth quadrants.

Hence, we see that the two normal coördinates for this case are **normal to each other.** We pointed out in an earlier chapter that the normal coördinates of a network were in general *not* normal to each other. Now we see that for two tuned-coupled circuits they are normal. In general it can be shown that for any number of meshes in a chain of tuned circuits, the normal coördinates form an orthogonal system.

For our two-mesh case we see that the transient current amplitudes for either mode will be the same in both meshes. We continue now with the determination of the complete solution. By (832) the equations (834a) and (834b) become:

$$\left.\begin{aligned} p^2 + \frac{R}{L+M}p + \frac{1}{(L+M)C} &= 0 \\ \text{and:} \qquad p^2 + \frac{R}{L-M}p + \frac{1}{(L-M)C} &= 0. \end{aligned}\right\} \tag{841}$$

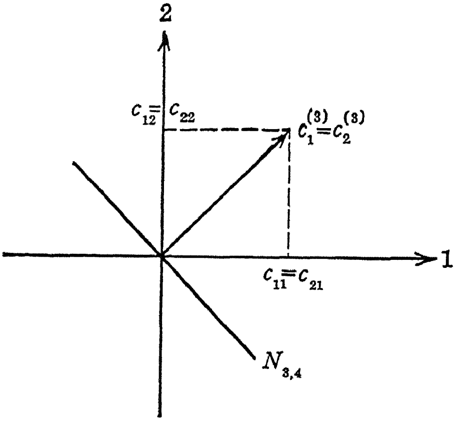


FIG. 124

Their solutions are given by:

$$p_{1,2} = -\frac{R}{2\,(L+M)} \pm j\sqrt{\frac{1}{(L+M)C} - \frac{R^2}{4\,(L+M)^2}} \tag{842}$$

and:

$$p_{3,4} = -\frac{R}{2\,(L-M)} \pm j\sqrt{\frac{1}{(L-M)C} - \frac{R^2}{4\,(L-M)^2}}. \tag{843}$$

If we write:

$$\left.\begin{aligned} p_{1,2} &= -\alpha_1 \pm j\omega_1 \\ \text{and:} \qquad p_{3,4} &= -\alpha_2 \pm j\omega_2 \end{aligned}\right\} \tag{844}$$

then the complete solutions take the form:

$$i_1 = (J_1^{(1)} e^{j\omega_1 t} + J_1^{(2)} e^{-j\omega_1 t})e^{-\alpha_1 t} + (J_1^{(3)} e^{j\omega_2 t} + J_1^{(4)} e^{-j\omega_2 t})e^{-\alpha_2 t} \tag{845}$$

and:

$$i_2 = (J_2^{(1)} e^{j\omega_1 t} + J_2^{(2)} e^{-j\omega_1 t})e^{-\alpha_1 t} + (J_2^{(3)} e^{j\omega_2 t} + J_2^{(4)} e^{-j\omega_2 t})e^{-\alpha_2 t} \tag{846}$$

where the steady-state terms are zero on account of the constant impressed force. We shall use the direct method for evaluating the integration constants. Also, we shall treat first the dissipationless case, i.e., that for neglected resistance. Then the decrements α_1 and α_2 in the above expressions become zero.

We therefore write (845) and (846) in the form:

$$\begin{aligned} i_1 &= C_{11}^{(1)}G^{(1)}e^{j\omega_1 t} + C_{11}^{(2)}G^{(2)}e^{-j\omega_1 t} + C_{11}^{(3)}G^{(3)}e^{j\omega_2 t} + C_{11}^{(4)}G^{(4)}e^{-j\omega_2 t} \\ i_2 &= C_{12}^{(1)}G^{(1)}e^{j\omega_1 t} + C_{12}^{(2)}G^{(2)}e^{-j\omega_1 t} + C_{12}^{(3)}G^{(3)}e^{j\omega_2 t} + C_{12}^{(4)}G^{(4)}e^{-j\omega_2 t}. \end{aligned} \tag{847}$$

The corresponding charge equations are found by integrating the above and adding the steady-state charge to the expression for the charge in mesh #1. Thus:

$$\left.\begin{aligned} q_1 &= \frac{C_{11}^{(1)}G^{(1)}e^{j\omega_1 t}}{j\omega_1} + \frac{C_{11}^{(2)}G^{(2)}e^{-j\omega_1 t}}{-j\omega_1} \\ &\qquad + \frac{C_{11}^{(3)}G^{(3)}e^{j\omega_2 t}}{j\omega_2} + \frac{C_{11}^{(4)}G^{(4)}e^{-j\omega_2 t}}{-j\omega_2} + EC \\ q_2 &= \frac{C_{12}^{(1)}G^{(1)}e^{j\omega_1 t}}{j\omega_1} + \frac{C_{12}^{(2)}G^{(2)}e^{-j\omega_1 t}}{-j\omega_1} \\ &\qquad + \frac{C_{12}^{(3)}G^{(3)}e^{j\omega_2 t}}{j\omega_2} + \frac{C_{12}^{(4)}G^{(4)}e^{-j\omega_2 t}}{-j\omega_2}. \end{aligned}\right\} \tag{848}$$

If we assume initial rest conditions, then we have for the determination of $G^{(1)} \ldots G^{(4)}$:

$$\left.\begin{aligned} C_{11}^{(1)}G^{(1)} + C_{11}^{(2)}G^{(2)} + C_{11}^{(3)}G^{(3)} + C_{11}^{(4)}G^{(4)} &= 0 \\ C_{12}^{(1)}G^{(1)} + C_{12}^{(2)}G^{(2)} + C_{12}^{(3)}G^{(3)} + C_{12}^{(4)}G^{(4)} &= 0 \\ \frac{C_{11}^{(1)}G^{(1)}}{j\omega_1} + \frac{C_{11}^{(2)}G^{(2)}}{-j\omega_1} + \frac{C_{11}^{(3)}G^{(3)}}{j\omega_2} + \frac{C_{11}^{(4)}G^{(4)}}{-j\omega_2} &= -EC \\ \frac{C_{12}^{(1)}G^{(1)}}{j\omega_1} + \frac{C_{12}^{(2)}G^{(2)}}{-j\omega_1} + \frac{C_{12}^{(3)}G^{(3)}}{j\omega_2} + \frac{C_{12}^{(4)}G^{(4)}}{-j\omega_2} &= 0. \end{aligned}\right\} \tag{849}$$

Now for the modes $p_{1,2}$ we have:

$$\left.\begin{aligned} &C_{11}^{(1)} = c_{22}^{(1)} = c_{11}^{(1)} = -c_{12}^{(1)} = -jM\omega_1 \qquad C_{11}^{(2)} = jM\omega_1 \\ &C_{12}^{(1)} = -c_{12}^{(1)} = -jM\omega_1 \qquad C_{12}^{(2)} = jM\omega_1 \end{aligned}\right\} \tag{850}$$

and for the modes $p_{3,4}$ we get:

$$\left.\begin{aligned} &C_{11}^{(3)} = c_{22}^{(3)} = c_{11}^{(3)} = c_{12}^{(3)} = jM\omega_2 \qquad C_{11}^{(4)} = -jM\omega_2 \\ &C_{12}^{(3)} = -c_{12}^{(3)} = -jM\omega_2 \qquad C_{12}^{(4)} = jM\omega_2. \end{aligned}\right\} \tag{851}$$

The system (849), therefore, becomes after some cancelling:

$$\left.\begin{aligned} -\omega_1 G^{(1)} + \omega_1 G^{(2)} + \omega_2 G^{(3)} - \omega_2 G^{(4)} &= 0 \\ -\omega_1 G^{(1)} + \omega_1 G^{(2)} - \omega_2 G^{(3)} + \omega_2 G^{(4)} &= 0 \\ - G^{(1)} - G^{(2)} + G^{(3)} + G^{(4)} &= \frac{EC}{M} \\ - G^{(1)} - G^{(2)} - G^{(3)} - G^{(4)} &= 0. \end{aligned}\right\} \tag{852}$$

This system is easily solved. We get:

$$\left.\begin{aligned} G^{(1)} = G^{(2)} &= -\frac{EC}{4M} \\ \text{and:} \qquad G^{(3)} = G^{(4)} &= +\frac{EC}{4M}. \end{aligned}\right\} \tag{853}$$

Hence the solutions (847) for the dissipationless case are:

$$\left.\begin{aligned} i_1 &= \frac{EC}{2}(\omega_1 \sin \omega_1 t + \omega_2 \sin \omega_2 t) \\ i_2 &= -\frac{EC}{2}(\omega_1 \sin \omega_1 t - \omega_2 \sin \omega_2 t). \end{aligned}\right\} \tag{854}$$

The amplitudes in both meshes are the same in magnitude, as predicted by the normal coördinate investigation above. Since:

$$\left.\begin{aligned} \omega_1 &= \frac{1}{\sqrt{(L+M)C}} \\ \omega_2 &= \frac{1}{\sqrt{(L-M)C}} \end{aligned}\right\} \tag{855}$$

we see that:

$$\left.\begin{aligned} \omega_2 &\to \omega_1 \\ \text{as:} \qquad M &\to 0. \end{aligned}\right\} \tag{856}$$

Thus for zero mutual:

$$\left.\begin{aligned} i_2 &= 0 \\ i_1 &= EC\omega_1 \sin \omega_1 t \end{aligned}\right\} \tag{857}$$

as it should.

In general the mutual inductance will be small, i.e.:

$$\frac{M}{L} = 0.1 \text{ or less.} \tag{858}$$

We shall discuss the result (854) under this condition. We then have by (855):

$$\left.\begin{aligned}\omega_1 &= \omega_0(1 - \delta)\\ \omega_2 &= \omega_0(1 + \delta)\end{aligned}\right\} \tag{859}$$

where:

$$\left.\begin{aligned}\omega_0 &= \frac{1}{\sqrt{LC}}\\ \delta &= \frac{M}{2L}\end{aligned}\right\} \tag{860}$$

and higher powers of δ are neglected in expanding the square root. By trigonometry we may then transform (854) into:

$$\left.\begin{aligned}i_1 &= EC\omega_0(\sin \omega_0 t \cdot \cos \delta\omega_0 t + \delta \cos \omega_0 t \cdot \sin \delta\omega_0 t)\\ i_2 &= EC\omega_0(\cos \omega_0 t \cdot \sin \delta\omega_0 t + \delta \sin \omega_0 t \cdot \cos \delta\omega_0 t)\end{aligned}\right\} \tag{861}$$

or approximately:

$$\left.\begin{aligned}i_1 &= EC\omega_0 \sin \omega_0 t \cdot \cos \delta\omega_0 t\\ i_2 &= EC\omega_0 \cos \omega_0 t \cdot \sin \delta\omega_0 t.\end{aligned}\right\} \tag{862}$$

These results are very interesting. We shall discuss them by means of the numerical data:

$$\left.\begin{aligned}L &= 10^{-3}\\ C &= 10^{-9}\\ M &= 10^{-4}\end{aligned}\right\}$$

which is the same as (787) except for $R = 0$. Then:

$$\left.\begin{aligned}\omega_0 &= 10^6\\ \delta &= 0.05\end{aligned}\right\} \tag{863}$$

so that:

$$\left.\begin{aligned}i_1 &= E \cdot 10^{-3} \sin 10^6 t \cos 5 \cdot 10^4 t\\ i_2 &= E \cdot 10^{-3} \cos 10^6 t \sin 5 \cdot 10^4 t.\end{aligned}\right\} \tag{864}$$

This is illustrated in Fig. 125. We see that the low frequency oscillation acts as an envelope for the higher frequency. The important point is the 90° phase displacement between the envelopes for the two currents. Thus, we see that the high-frequency oscillation has a large amplitude in mesh #1 when it is small in mesh #2 and vice versa. This is due to the fact that the meshes alternately feed the energy of the system back and forth.

First #1 has it; then it is all fed over into #2 at the first node of i_1 or the first anti-node of i_2, etc. As the mutual inductance decreases, the period of energy interplay, i.e., of the envelopes, becomes longer. Finally, for $M = 0$ this period becomes infinite, so that the energy never gets over into mesh #2.

The solution of this network with resistance taken into account proceeds in exactly the same manner as above, except that the complex roots (842) and (843) are used instead of merely their imaginary parts. The result may also be put in an approximate form for the case of small mutual. There results:

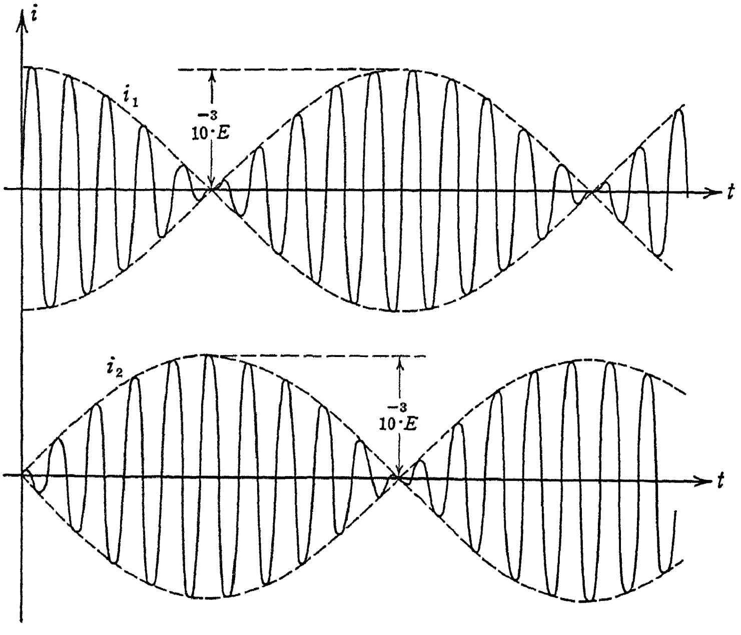


FIG. 125

$$\left.\begin{aligned} i_1 &= e^{-\alpha_0 t} E C \omega_0 \sin \omega_0 t \cdot \cos \delta\omega_0 t \\ i_2 &= e^{-\alpha_0 t} E C \omega_0 \cos \omega_0 t \cdot \sin \delta\omega_0 t \end{aligned}\right\} \tag{865}$$

where:

$$\alpha_0 = \frac{R}{2L} \tag{866}$$

the difference in the two decrements being too small to have any appreciable effect.

When the transient behavior of this network is studied with a harmonic impressed force, the above interplay of energy, which is so characteristic of the natural behavior of this system, is submerged in a multitude of additional effects which depend upon the relative magnitudes of the impressed and natural frequencies, the phase of the voltage at the switching instant, etc. Many interesting points are involved in such a study, but we shall not take them up here.

We rather prefer to continue with the consideration of three inductively coupled tuned circuits as illustrated in Fig. 120. Again we shall treat the application of a constant force. Then we have the following conditions to satisfy:

$$\left.\begin{aligned} c_{11}J_1 + c_{12}J_2 + 0 \quad\;\; &= 0 \\ c_{21}J_1 + c_{22}J_2 + c_{23}J_3 &= 0 \\ 0 \quad\;\; + c_{32}J_2 + c_{33}J_3 &= 0. \end{aligned}\right\} \tag{867}$$

Here we have:

$$\left.\begin{aligned} c_{11} = c_{22} = c_{33} &= Lp + R + \frac{1}{Cp} \\ c_{12} = c_{21} = c_{23} = c_{32} &= Mp. \end{aligned}\right\} \tag{868}$$

Therefore the modular determinant becomes:

$$D(p) = \begin{vmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{vmatrix} = c_{11}(c_{11} - \sqrt{2}\,c_{12})\,(c_{11} + \sqrt{2}\,c_{12}). \tag{869}$$

The determinantal equation is factorable. We have:

$$\left.\begin{aligned} pc_{11} &= 0 \\ p(c_{11} - \sqrt{2}\,c_{12}) &= 0 \\ p(c_{11} + \sqrt{2}\,c_{12}) &= 0 \end{aligned}\right\} \tag{870}$$

or by (868):

$$\left.\begin{aligned} &p^2 + \frac{R}{L}p + \frac{1}{LC} = 0 \\ &p^2 + \frac{R}{L - \sqrt{2}\,M}p + \frac{1}{(L - \sqrt{2}\,M)C} = 0 \\ &p^2 + \frac{R}{L + \sqrt{2}\,M}p + \frac{1}{(L + \sqrt{2}\,M)C} = 0. \end{aligned}\right\} \tag{870a}$$

Let us denote the roots of these quadratics by:

$$\left.\begin{aligned} p_{1,2} &= -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \\ p_{3,4} &= -\frac{R}{2(L-\sqrt{2}M)} \pm j\sqrt{\frac{1}{(L-\sqrt{2}M)C} - \frac{R^2}{4(L-\sqrt{2}M)^2}} \\ p_{5,6} &= -\frac{R}{2(L+\sqrt{2}M)} \pm j\sqrt{\frac{1}{(L+\sqrt{2}M)C} - \frac{R^2}{4(L+\sqrt{2}M)^2}} \end{aligned}\right\} \tag{871}$$

respectively. More briefly we will write:

$$\left.\begin{aligned} p_{1,2} &= -\alpha_1 \pm j\omega_1 \\ p_{3,4} &= -\alpha_2 \pm j\omega_2 \\ p_{5,6} &= -\alpha_3 \pm j\omega_3. \end{aligned}\right\} \tag{872}$$

Since the steady-state currents are zero, we have for the complete current solutions:

$$\left.\begin{aligned} i_1 &= \sum_{\nu=1}^{6} C_{11}^{(\nu)} G^{(\nu)} e^{p_\nu t} \\ i_2 &= \sum_{\nu=1}^{6} C_{12}^{(\nu)} G^{(\nu)} e^{p_\nu t} \\ i_3 &= \sum_{\nu=1}^{6} C_{13}^{(\nu)} G^{(\nu)} e^{p_\nu t}. \end{aligned}\right\} \tag{873}$$

For the mode p_1 we have the array of c_{ik}'s:

$$\begin{Vmatrix} 0 & c_{12}^{(1)} & 0 \\ c_{12}^{(1)} & 0 & c_{12}^{(1)} \\ 0 & c_{12}^{(1)} & 0 \end{Vmatrix} \tag{874}$$

from which:

$$\left.\begin{aligned} C_{11}^{(1)} &= -[c_{12}^{(1)}]^2 \\ C_{12}^{(1)} &= 0 \\ C_{13}^{(1)} &= [c_{12}^{(1)}]^2. \end{aligned}\right\} \tag{875}$$

For the conjugate mode we get the conjugates of the above values. For the mode p_3 we have the array:

$$\begin{Vmatrix} \sqrt{2}\,c_{12}^{(3)} & c_{12}^{(3)} & 0 \\ c_{12}^{(3)} & \sqrt{2}\,c_{12}^{(3)} & c_{12}^{(3)} \\ 0 & c_{12}^{(3)} & \sqrt{2}\,c_{12}^{(3)} \end{Vmatrix} \tag{876}$$

so that:

$$\left.\begin{aligned} C_{11}^{(3)} &= [c_{12}^{(3)}]^2 \\ C_{12}^{(3)} &= -\sqrt{2}\,[c_{12}^{(3)}]^2 \\ C_{13}^{(3)} &= [c_{12}^{(3)}]^2. \end{aligned}\right\} \tag{877}$$

The conjugate values result for p_4. Similarly, we have for p_5:

$$\left\| \begin{matrix} -\sqrt{2}\, c_{12}^{(5)} & c_{12}^{(5)} & 0 \\ c_{12}^{(5)} & -\sqrt{2}\, c_{12}^{(5)} & c_{12}^{(5)} \\ 0 & c_{12}^{(5)} & -\sqrt{2}\, c_{12}^{(5)} \end{matrix} \right\| \tag{878}$$

Hence:

$$\left.\begin{aligned} C_{11}^{(5)} &= [c_{12}^{(5)}]^2 \\ C_{12}^{(5)} &= \sqrt{2}[c_{12}^{(5)}]^2 \\ C_{13}^{(5)} &= [c_{12}^{(5)}]^2. \end{aligned}\right\} \tag{879}$$

For p_6 the conjugate values are obtained. In order to evaluate the minors given by (875), (877), and (879) we have merely to calculate the value of:

$$c_{12}^{(\nu)} = Mp_\nu \tag{880}$$

which is a very simple matter. The reason for this simplicity is due to the uniformity of the network, i.e., the fact that all the meshes are alike, and all the coupling terms are alike. This makes it possible to express everything in terms of the coefficient c_{12}.

We shall continue with the evaluation of the integration constants for the dissipationless case, i.e., for neglected resistance. The procedure is the same whether resistance is considered or not, except that the analytic work becomes somewhat heavy when resistances are taken into account. We could, of course, resort to a numerical example. But we believe that an analytic solution is more instructive. In order to disemburden this presentation as much as possible, we, therefore, neglect the resistances at this point. Then we have:

$$\left.\begin{aligned} p_{1,2} &= \pm j\omega_1 \\ p_{3,4} &= \pm j\omega_2 \\ p_{5,6} &= \pm j\omega_3 \end{aligned}\right\} \tag{872a}$$

with:

$$\left.\begin{aligned} \omega_1 &= \frac{1}{\sqrt{LC}} \\ \omega_2 &= \frac{1}{\sqrt{(L - \sqrt{2}\,M)C}} \\ \omega_3 &= \frac{1}{\sqrt{(L + \sqrt{2}\,M)C}}. \end{aligned}\right\} \tag{881}$$

Thus the minors (875), (877), and (879) become:

$$\left.\begin{aligned} C_{11}{}^{(1)} &= M\omega_1{}^2 \\ C_{12}{}^{(1)} &= 0 \\ C_{13}{}^{(1)} &= -M\omega_1{}^2 \end{aligned}\right\} \qquad (875a)$$

$$\left.\begin{aligned} C_{11}{}^{(3)} &= -M\omega_2{}^2 \\ C_{12}{}^{(3)} &= \sqrt{2}\,M\omega_2{}^2 \\ C_{13}{}^{(3)} &= -M\omega_2{}^2 \end{aligned}\right\} \qquad (877a)$$

and:

$$\left.\begin{aligned} C_{11}{}^{(5)} &= -M\omega_3{}^2 \\ C_{12}{}^{(5)} &= -\sqrt{2}\,M\omega_3{}^2 \\ C_{13}{}^{(5)} &= -M\omega_3{}^2. \end{aligned}\right\} \qquad (879a)$$

Note that these are all real quantities. Hence, their conjugates are the same values.

Before we are able to evaluate the integration constants, we must determine the charge equations by integrating the set (873). We get:

$$\left.\begin{aligned} q_1 &= \sum_{\nu=1}^{6} \frac{C_{11}{}^{(\nu)}G^{(\nu)}}{p_\nu} e^{p_\nu t} + EC \\ q_2 &= \sum_{\nu=1}^{6} \frac{C_{12}{}^{(\nu)}G^{(\nu)}}{p_\nu} e^{p_\nu t} \\ q_3 &= \sum_{\nu=1}^{6} \frac{C_{13}{}^{(\nu)}G^{(\nu)}}{p_\nu} e^{p_\nu t}. \end{aligned}\right\} \qquad (882)$$

For initial rest conditions we, therefore, have the following system to solve, which we obtain after some cancellation:

$$\left.\begin{aligned} \omega_1{}^2(G^{(1)}+G^{(2)}) - \omega_2{}^2(G^{(3)}+G^{(4)}) - \omega_3{}^2(G^{(5)}+G^{(6)}) &= 0 \\ 0 \qquad + \omega_2{}^2(G^{(3)}+G^{(4)}) - \omega_3{}^2(G^{(5)}+G^{(6)}) &= 0 \\ \omega_1{}^2(G^{(1)}+G^{(2)}) + \omega_2{}^2(G^{(3)}+G^{(4)}) + \omega_3{}^2(G^{(5)}+G^{(6)}) &= 0 \\ \omega_1(G^{(1)}-G^{(2)}) - \omega_2(G^{(3)}-G^{(4)}) - \omega_3(G^{(5)}-G^{(6)}) &= \frac{-jEC}{M} \\ 0 \qquad + \omega_2(G^{(3)}-G^{(4)}) - \omega_3(G^{(5)}-G^{(6)}) &= 0 \\ \omega_1(G^{(1)}-G^{(2)}) + \omega_2(G^{(3)}-G^{(4)}) + \omega_3(G^{(5)}-G^{(6)} &= 0. \end{aligned}\right\} \qquad (883)$$

This system looks very formidable, but it is quite simple to solve. The first three equations are in terms of the sums of pairs of unknowns, while the second three are in terms of the corresponding differences. If we consider the sums as unknowns, then the first three equations form a homogeneous group. The determinant of this group does not vanish, however, so that we have:

$$\left.\begin{aligned} G^{(1)} + G^{(2)} &= 0 \\ G^{(3)} + G^{(4)} &= 0 \\ G^{(5)} + G^{(6)} &= 0. \end{aligned}\right\} \tag{884}$$

The second three equations form an inhomogeneous group whose solutions are found in the usual manner. They are:

$$\left.\begin{aligned} G^{(1)} - G^{(2)} &= \frac{-jEC}{2\,M\omega_1} \\ G^{(3)} - G^{(4)} &= \frac{jEC}{4\,M\omega_2} \\ G^{(5)} - G^{(6)} &= \frac{jEC}{4\,M\omega_3}. \end{aligned}\right\} \tag{885}$$

Hence:

$$\left.\begin{aligned} G^{(1)} &= \frac{-jEC}{4\,M\omega_1} & G^{(2)} &= \frac{jEC}{4\,M\omega_1} \\ G^{(3)} &= \frac{jEC}{8\,M\omega_2} & G^{(4)} &= \frac{-jEC}{8\,M\omega_2} \\ G^{(5)} &= \frac{jEC}{8\,M\omega_3} & G^{(6)} &= \frac{-jEC}{8\,M\omega_3}. \end{aligned}\right\} \tag{886}$$

Finally, the current solutions become:

$$\left.\begin{aligned} i_1 &= \frac{EC}{2}\left(\omega_1 \sin \omega_1 t + \frac{\omega_2}{2} \sin \omega_2 t + \frac{\omega_3}{2} \sin \omega_3 t\right) \\ i_2 &= \frac{-EC}{2\sqrt{2}}(\omega_2 \sin \omega_2 t - \omega_3 \sin \omega_3 t) \\ i_3 &= \frac{-EC}{2}\left(\omega_1 \sin \omega_1 t - \frac{\omega_2}{2} \sin \omega_2 t - \frac{\omega_3}{2} \sin \omega_3 t\right). \end{aligned}\right\} \tag{887}$$

Note that the frequency ω_1 does not appear in mesh #2. This is due to the fact that the corresponding normal coördinate lies in the 1–3 plane, i.e., it has no component in the axis 2, as may be seen from (875*a*). These minors, incidentally, may be looked upon as being components of a vector which coincides with the respective normal coördinate for which the minors are determined. If the normal coördinates are at right angles to each other, then the scalar products of these vectors must vanish. That is:

$$\left.\begin{aligned} C_{11}^{(1)}C_{11}^{(3)} + C_{12}^{(1)}C_{12}^{(3)} + C_{13}^{(1)}C_{13}^{(3)} &= 0 \\ C_{11}^{(1)}C_{11}^{(5)} + C_{12}^{(1)}C_{12}^{(5)} + C_{13}^{(1)}C_{13}^{(5)} &= 0 \\ C_{11}^{(3)}C_{11}^{(5)} + C_{12}^{(3)}C_{12}^{(5)} + C_{13}^{(3)}C_{13}^{(5)} &= 0. \end{aligned}\right\} \tag{888}$$

By (875*a*), (877*a*), and (879*a*) we see that this is correct. Hence, the normal coördinates again form an orthogonal system.

It is an interesting fact that modes may be suppressed in the fashion indicated by (887) for mesh #2. We pointed out earlier that certain modes may be suppressed by causing one or more of the actual meshes to become normal coördinates. Now we see that suppression of modes may occur without having any normal meshes in the network. In the case of a chain of identical coupled circuits such suppression occurs whenever the modular determinant contains the common factor c_{11} as shown by (869). It is rather peculiar that the frequency ω_1 should be fed from mesh #1 into mesh #3 without appearing in the intermediate mesh #2. This is just one of the many extraordinary tricks that networks may play.

Note also that for meshes #1 and #3, the amplitude of the frequency ω_1 is twice the amplitudes for ω_2 and ω_3. In the steady-state treatment above, we also found this same relationship for the corresponding resonance maxima, at least as an asymptotic value. It is interesting to see that the same relationship of amplitudes exists in the transient state. In general this may be explained on the basis of the excitation of normal coördinates. The latter govern the behavior in the steady state as well as in the transient state. Much still remains to be done in the way of correlating steady state with transient behavior in electrical networks. The writer believes that real progress along these lines must proceed through a more thorough study of the excitation of normal modes and their respective normal coördinates. The concept of normal coördinates in electrical networks is the only means whereby we can arrive at a deeper understanding of the mechanism of network behavior. It is for this reason that the subject of normal coördinates has received so much attention throughout the above presentation.

The result (887) may be discussed for the case of small values of mutual inductance, just as this was done for the two-mesh example above. Here we have:

$$\left.\begin{aligned} \omega_1 &= \frac{1}{\sqrt{LC}} = \omega_0 \\ \omega_2 &= \omega_0(1 + \delta) \\ \omega_3 &= \omega_0(1 - \delta) \end{aligned}\right\} \tag{889}$$

with:

$$\delta = \frac{M}{\sqrt{2}\, L}\,. \tag{890}$$

Then by trigonometry we get:

$$\left.\begin{aligned} i_1 &\cong EC\omega_0 \sin \omega_0 t \cdot \frac{1 + \cos \delta\omega_0 t}{2} \\ i_2 &\cong -\frac{EC\omega_0}{\sqrt{2}} \cos \omega_0 t \cdot \sin \delta\omega_0 t \\ i_3 &\cong -EC\omega_0 \sin \omega_0 t \cdot \frac{1 - \cos \delta\omega_0 t}{2}\,. \end{aligned}\right\} \tag{891}$$

This result is illustrated in Fig. 126. Now the envelopes for the currents in the first and third meshes look different from those in the two-mesh case, but the current i_2 is substantially the same except for the fact that its amplitude is smaller. The general effect is an interplay of energy between the three meshes. As the mutual is decreased, the period of this interplay lengthens. Finally, for zero mutual, the period becomes infinite, so that no energy interchange takes place.

When the system is slightly damped, the general appearance of the behavior will be the same except for an exponential decay in all three meshes, so that the entire process comes to rest after a sufficient elapse of time.

In the present example the direct method of evaluating the integration constants is much shorter than the method by means of successive initial derivatives. This is due to the simple relations which hold among the components of the modular vector groups.

5. Uniform Structures. The above discussion of tuned-coupled circuits touches upon an extensive field of network theory which is of considerable importance in communication work. Very much the same general ideas as brought out above, are found also when other types of coupling are employed between the meshes. Indeed, the meshes themselves need not be oscillatory in order to possess some of the above characteristics relative to normal modes and normal coördinates. The chief characteristic of the above networks is the uniformity of their structure, — the fact that each succeeding mesh is the same as the one preceding it, and that all the common branches are identical. Fig. 127 illustrates the most general form for this type of network, for n meshes. We call such a network a **uniform recurrent structure of the ladder type.** The component impedances z_1 and z_2 may have any structure at all, so long as they are the same in all the branches, as indicated.

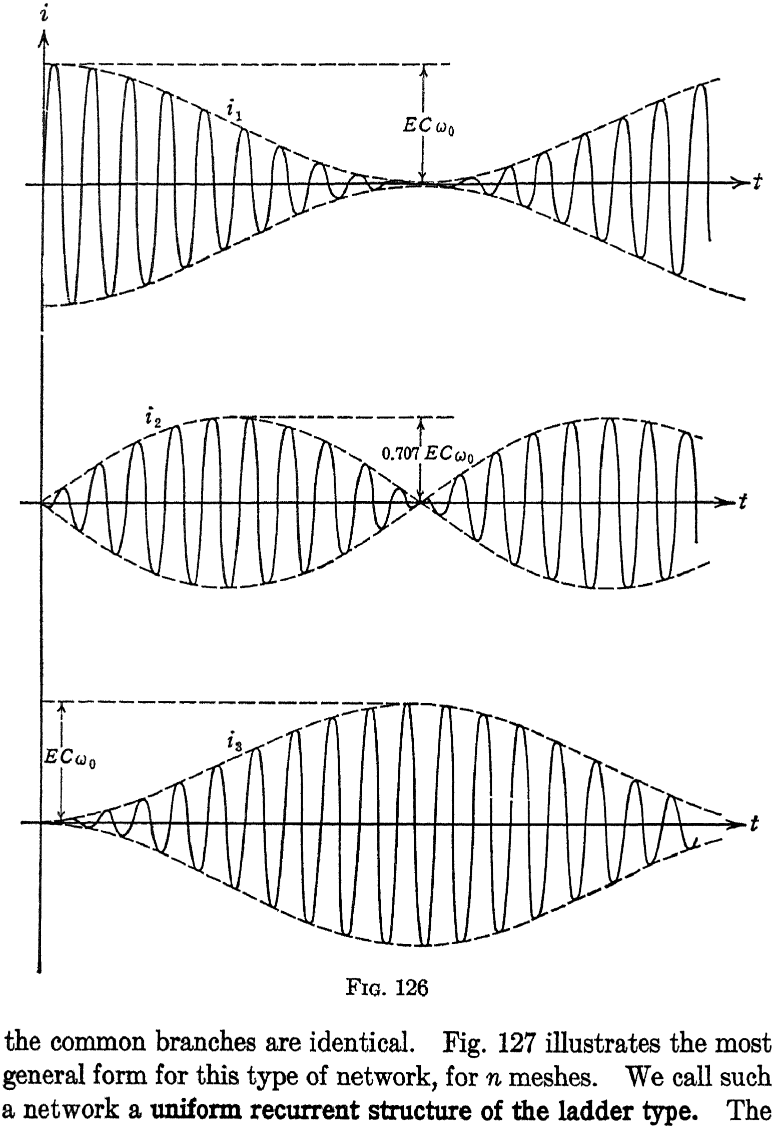


Fig. 126

Such uniform structures form the basis for the so-called artificial line theory, to which we shall come in the second volume of this presentation. They also form the basis for the theory of operation and design of one type of electric wave filter. The latter is essentially a four-terminal network which possesses selective properties regarding frequency. The tuned circuits investigated above may be looked upon in a sense as being wave filters of this type. We shall have more to say on this point in another chapter which we shall devote exclusively to the subject of frequency selection. However, it is well for the student to recognize at this time that the basic principle underlying all selective networks is the simple resonance phenomenon. So-called pass-bands are nothing more than resonance regions for the network. This was clearly illustrated above in the case of tuned circuits.

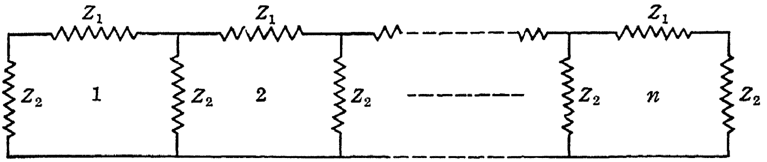


FIG. 127

The study of selective networks, therefore, becomes an investigation of the distribution of the normal modes of certain types of networks.

In this connection the study of the geometric orientation of normal coördinates also promises to be a fruitful aid. We saw that certain orientations gave rise to the suppression of normal modes in certain meshes. Broadly interpreted, this is nothing more than frequency discrimination also. The fact that the normal coördinates for uniform recurrent structures form an orthogonal system, is a step in the direction of the promotion of this concept. However, the bulk of such investigations still remains to be done, so that this field is a promising one for further research along the lines of selective networks.

In concluding this chapter we feel that the number of problems that have been treated are few compared to the vast supply from which we could choose. However, our object here was merely to give the reader a more concrete idea of the utility of the fundamental methods of attack which were outlined in the preceding chapters. Now that he has seen how to apply the methods, he may continue on his own initiative.

PROBLEMS TO CHAPTER VIII

8–1. A transformer consists of four identical windings. Assuming that the mutual inductances are all equal, determine the steady-state short circuit currents in three of the windings when a harmonic force is impressed in the fourth.

8–2. Determine the transient currents in three short-circuited windings of the transformer of Prob. **8–1** when a constant force is suddenly impressed in the fourth.

8–3. Repeat Prob. **8–2** for a harmonic impressed force: (*a*) for the case where the voltage is passing through zero and increasing positively at the switching instant, and (*b*) for the case where the voltage at that instant is passing through its positive maximum.

8–4. Determine the steady-state selective properties of a chain of four tuned-coupled circuits for which the mutual inductance between the second and third is much smaller than that between the first and second, and the third and fourth.

8–5. A network consists of four meshes. Mesh #1 consists of three identical coils in series, each having resistance and inductance. The other three meshes are identical and highly oscillatory, each containing the same amount of resistance, inductance, and capacitance. Mesh #1 is individually coupled to each of the other three by purely mutual inductance through the three coils respectively. The other meshes have no coupling relative to each other. A harmonic generator is impressed in mesh #1. Determine the steady-state resonance phenomena in this network.

8–6. Determine the components of the modular vector group, and on the basis of these check the results of Prob. **8–5** regarding the relative magnitudes of the resonance maxima.

8–7. An impedance consists of a coil and condenser in parallel. The coil has resistance and inductance. Plot the real and imaginary parts of this impedance, as well as its magnitude as functions of frequency. Use the numerical values:

$$R = 2 \qquad L = 10^{-3} \qquad C = 10^{-9}.$$

Add a resistance of 10^5 ohms in parallel with this impedance and repeat the above plots for the resulting combination.

8–8. Two tuned-coupled circuits are characterized by the parameters:

$$R = 10 \qquad L = \frac{1}{4\pi^2} \qquad C = 10^{-6}.$$

With a harmonic generator of constant amplitude impressed in the first mesh, plot the steady-state current in the second as a function of frequency over a range which includes the resonance region, for the following values of mutual inductance:

$$M = \frac{L}{5} \qquad M = \frac{L}{10} \qquad M = \frac{L}{100}.$$

8–9. Repeat Prob. **8–8** for the parameters:

$$R = 5 \qquad L = 10^{-3} \qquad C = 10^{-9}.$$

8–10. A network is characterized by the following parameters:

$$L_{11} = L_{22} = 10^{-2} \qquad S_{11} = S_{22} = 2 \cdot 10^6 \qquad S_{12} = 10^6.$$

Construct the network. For a suddenly impressed constant force in the first mesh, determine the complete current solution in both meshes by the classical method assuming initial rest conditions, and using the direct method for evaluating the constants of integration.

8–11. Repeat Prob. **8–10** by Heaviside's formula and check.

8–12. Determine the geometric positions of the normal coördinates for the network of Prob. **8–10**, and check the relative magnitudes of the transient current amplitudes as determined previously in Probs. **8–10** and **8–11**. Show that the normal coördinates form an orthogonal system.

8–13. Repeat Prob. **8–10** with the additional data:

$$R_{11} = R_{22} = 10.$$

8–14. Repeat Prob. **8–11** with the additional data of Prob. **8–13**.

8–15. Determine the geometric positions of the normal coördinates for the network of Probs. **8–13** and **8–14**, and check the relative magnitudes of the transient current amplitudes as given by the solutions to those problems. Also predict the relative magnitudes of the steady-state resonance phenomena for this network.

8–16. Determine the steady-state solution to the network of Prob. **8–10** with the additional data of Prob. **8–13**, and plot against frequency over a band which includes the resonance region, first with a harmonic generator of constant amplitude in mesh #1, and second, with this source in mesh #2.

8–17. Repeat Prob. **8–10** for the three-mesh network characterized by:

$$L_{11} = L_{22} = L_{33} = 10^{-2} \qquad S_{11} = S_{22} = S_{33} = 2 \cdot 10^6 \qquad S_{12} = S_{23} = 10^6.$$

8–18. Repeat Prob. **8–11** using the data of Prob. **8–17**.

8–19. Repeat Prob. **8–12**, using the data of Prob. **8–17**.

8–20. Repeat Prob. **8–17** with the additional data:

$$R_{11} = R_{22} = R_{33} = 10.$$

8–21. Determine the geometric positions of the normal coördinates for the network of Prob. **8–17** with the additional data of Prob. **8–20**, and predict the character of the resonance phenomena in the steady state. Check these by the usual method assuming a harmonic generator successively in the first, second, and third mesh.

CHAPTER IX

SOME IMPORTANT NETWORK FORMULAE

1. Heaviside's Expansion Formula. In this discussion no proof will be given for Heaviside's formula. Such proofs are numerous and may be found in a variety of forms.[1] They are interesting, and help materially in the understanding of the application of the formula to specific cases. A proof could not be given here because it presupposes too much in the way of mathematical equipment. The student may supply himself with this information at a later date, but he should be introduced to the subject at this time. That is, he should know of the existence of this formula, of its various forms, and of its utility. This is what we propose to take up here.

Heaviside's formula is not a short-cut method for obtaining the complete solution to an electrical network with specified excitation. The solution of a specific case by means of the formula involves practically the same amount of mechanical labor as the classical method outlined in the preceding chapters. It is not the object of the formula to afford a short-cut, although its compactness may easily lead one to believe that it accomplishes this end. Briefly, Heaviside's formula is just one more method whereby the integration constants in the complete solution may be evaluated. It is a very elegant method. It is an extremely compact method from the standpoint of analytic form. However, the process of obtaining a complete solution by means of it involves the determination of the steady-state solution in the usual fashion, the determination of the normal modes in the usual fashion, and the evaluation of the transient current amplitudes by means of the formula.

From this it would seem as though the utility of Heaviside's formula were not very great. From the practical standpoint this is so; and as a result it is not used very much in getting network solutions. However, analytically the situation is quite otherwise. Heaviside's formula is used quite frequently in the theoretical analysis of networks, i.e., in research work. There its usefulness is due almost entirely to the compactness of its form. We have been emphasizing at intervals throughout the above that the chief difficulty in network investigations lies in the unwieldiness of the expressions involved. This unwieldiness often obscures the vision to such an extent as to make it impossible to forge ahead. The solution to such situations is always found by making use of certain mathematical forms which are able to compress the bulk into a smaller space. In such cases Heaviside's formula is frequently very useful. Hence, a thorough understanding of it is worth acquiring.

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[1] For an excellent review of this situation see V. Bush, *Operational Circuit Analysis*, John Wiley & Sons, 1929, pp. 86 e.s.

Heaviside's formula masquerades in a greater variety of forms throughout the literature than almost any other of its kin. Perhaps the commonest form is the following:

$$\dot{i}_{sk} = \frac{E}{Z_{sk}(0)} + \sum_{\nu=1}^{m} \frac{Ee^{p_\nu t}}{p_\nu \cdot Z_{sk}'(p_\nu)} \cdot \tag{892}$$

Here E is the amplitude of a constant impressed force. Z_{sk} is the steady-state network impedance to be used for determining the current in mesh #s with the impressed force in mesh #k or vice versa, as given by eq. (328) of Chapter IV. Furthermore:

$$Z_{sk}'(p_\nu) = \left[\frac{dZ_{sk}(p)}{dp}\right]_{p=p_\nu} \tag{893}$$

where $Z_{sk}(p)$ is the function obtained by replacing $j\omega$ in Z_{sk} by p. Finally, p_ν is one of the m consecutively numbered modes. Note that m is not the number of meshes. The formula is to be used for finding the complete solution for the current in mesh #s with a constant voltage E suddenly impressed in mesh #k, or vice versa. We see that $Z_{sk}(0)$ is the steady-state network impedance for a zero frequency, i.e., a constant force. Hence the first term of (892) represents the steady-state solution. The second term is a sum of m terms, one for each mode. Altogether these m terms comprise the transient portion of the complete solution. In the preceding chapters we denoted the number of meshes by n, and thus, in general, obtained $2n$ terms in the transient solution. This is the same number of terms as is contained in the sum given by (892), for the general case where the number of modes equals twice the number of meshes.

The formula has one major restriction which should be clearly understood. It gives the solution on the basis of initial rest conditions. That is, it tacitly assumes that the initial mesh-currents and charges are zero. When the initial conditions are not rest conditions, the solution cannot be found by Heaviside's formula directly. The formula will then give only a part of the complete solution. In order to obtain the latter, we must add to the result given by the formula, the solution for zero impressed forces with the specified initial conditions. This part of the work must be carried out in the usual fashion as outlined in the previous chapters. The correctness of this procedure is due to the linearity of the network. Namely, when arbitrary initial conditions obtain, then the complete solution may be gotten by superposing linearly the solution for initial rest conditions with the specified impressed forces, and the solution for the specified initial conditions with zero-impressed forces under the same switching conditions. However, in a practical case there would be no sense in doing all this, because the total labor involved would be almost double the usual amount. Here again the utility of the formula is merely of an analytic character.

Perhaps it may be well, before proceeding with other forms, to give an illustration of the use of the formula. Suppose we consider the simple R, L, C circuit with initial rest conditions. Here the network impedance is:

$$Z = R + j\left(L\omega - \frac{1}{C\omega}\right). \tag{894}$$

The function $Z(p)$ is:

$$Z(p) = Lp + R + \frac{1}{Cp}. \tag{895}$$

The determinantal equation is:

$$p^2 + \frac{R}{L}p + \frac{1}{LC} = 0. \tag{896}$$

The modes are therefore:

$$\left.\begin{aligned} p_{1,2} &= -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \\ \text{or:} \qquad p_1 &= -\alpha + jg \\ p_2 &= -\alpha - jg \end{aligned}\right\} \tag{897}$$

for the oscillatory case. For the steady-state term of (892) we see that:

$$Z(0) = \infty \tag{898}$$

so that the steady state is zero as it should be. For the evaluation of the sum we have to form the derivative of $Z(p)$ with respect to p. Here we see that:

$$\left.\begin{aligned} Z(p) &= \frac{L}{p}\left(p^2 + \frac{R}{L}p + \frac{1}{LC}\right) \\ &= \frac{L}{p}(p - p_1)(p - p_2) \end{aligned}\right\} \tag{899}$$

because p_1 and p_2 are the roots of the polynomial in the parenthesis. Hence:

$$\frac{dZ(p)}{dp} = \frac{L}{p}[(p - p_1) + (p - p_2)] - \frac{L}{p^2}(p - p_1)(p - p_2). \tag{900}$$

From this it is easy to see that:

$$p_1\left(\frac{dZ(p)}{dp}\right)_{p=p_1} = L(p_1 - p_2) \tag{901}$$

and:

$$p_2\left(\frac{dZ(p)}{dp}\right)_{p=p_2} = L(p_2 - p_1). \tag{902}$$

Substitution into (892), therefore, gives:

$$i = 0 + \frac{E}{L}\left[\frac{e^{p_1t}}{p_1 - p_2} + \frac{e^{p_2t}}{p_2 - p_1}\right]$$

which evaluates to:

$$i = \frac{E}{Lg}e^{-\alpha t}\sin gt. \tag{903}$$

This we recognize as the correct result. It will be observed that the amount of mechanical motion involved in this process is about the same as for the usual method.

There are several other limitations to the formula besides the one regarding initial conditions. The more important of these is that the determinantal equation should have no equal roots, i.e., there should be no coincident modes. We can see from the form of (892) that this limitation exists. It has been shown how the formula may be altered in order to make it hold for coincident roots also, but this subject is hardly important enough to warrant discussion here. The proof for this extension of the formula rests upon the theory of integration in the complex plane, and although very interesting, is rather meaningless without a knowledge of function theory. It may be found in the literature as well as in several advanced texts on this subject. Coincident roots occur only in special cases. Their discussion in this connection is hardly fundamental enough.

Another limitation of Heaviside's formula is that the function:

$$pZ(p)$$

shall not reduce to a constant. This is a rather trivial limitation from the practical standpoint, because it never occurs in physical networks. It would occur if we considered a condenser all by itself, thus:

$$p \cdot \frac{1}{Cp} = \frac{1}{C}.$$

But this is obviously nonsense, because a condenser alone, without at least a resistance in series, is nothing physically realizable. Consequently, there is not much sense in worrying over this limitation.

The formula has also been extended to cover the case of a harmonic impressed force. In fact it can be extended to include any force function of the exponential form. These matters also belong in a more advanced treatment of network theory because they concern themselves with more specialized phases. We merely mention them here so that the student will know of their existence. He may be interested to follow them up at a later time.

We shall now concern ourselves with several other forms in which the formula may be written, so that the reader may more easily recognize these in case he is interested in reading some of the literature. In this connection we recognize that, since the network impedance is given by:

$$Z_{sk} = \frac{D}{B_{sk}} \tag{904}$$

where D is the steady-state network determinant and B_{sk} one of its principal minors, the function $Z(p)$ must be:

$$Z_{sk}(p) = \frac{D(p)}{C_{sk}} \tag{905}$$

where $D(p)$ is the modular determinant, and C_{sk} is one of its principal minors. By (905) we see that:

$$\frac{dZ_{sk}(p)}{dp} = \frac{C_{sk} \cdot \frac{dD(p)}{dp} - D(p) \cdot \frac{dC_{sk}}{dp}}{(C_{sk})^2}. \tag{906}$$

Now since any mode p_ν is that value of p for which:

$$D(p) = 0$$

we see that:

$$\left[\frac{dZ_{sk}(p)}{dp}\right]_{p=p_\nu} = \left[\frac{1}{C_{sk}} \cdot \frac{dD(p)}{dp}\right]_{p=p_\nu}. \tag{907}$$

This is also written:

$$Z_{sk}'(p_\nu) = \frac{D_\nu'}{C_{sk}^{(\nu)}}. \tag{907a}$$

Thus the formula (892) becomes:

$$i_{sk} = \frac{E}{Z_{sk}(0)} + E \sum_{\nu=1}^{m} \frac{C_{sk}^{(\nu)} e^{p_\nu t}}{p_\nu D_\nu'}. \tag{908}$$

Another form is obtained if we introduce the new function:

$$Z^*(p) = pZ(p). \tag{909}$$

Then we see that:

$$\frac{dZ^*}{dp} = p\frac{dZ}{dp} + Z. \tag{910}$$

But by (905):

$$Z(p_\nu) = Z_\nu = 0 \tag{911}$$

so that:

$$\left(\frac{dZ^*}{dp}\right)_{p=p_\nu} = Z_\nu^{*\prime} = p_\nu Z_\nu' \tag{912}$$

where primes are used to denote differentiation with respect to p. From (910) we also see that:

$$\left(\frac{dZ^*}{dp}\right)_{p=0} = Z^{*\prime}(0) = Z(0). \tag{913}$$

The two relations (912) and (913) suggest that the formula (892) be written as:

$$i = E\sum_{\nu=0}^{m} \frac{e^{p_\nu t}}{Z_\nu^{*\prime}} \qquad (914)$$

where the indices have been omitted to avoid confusion. In this new form the values p_ν are understood to be roots of the equation:

$$Z^* = pZ = 0. \qquad (915)$$

They are the modes of the system, plus the root $p = 0$ which accounts for the steady-state term in (914). This zero root will be present only when a steady-state solution exists. The form (914) thus includes the steady-state term in the sum. It also simplifies the denominator of each term in the sum in that the factor p_ν does not appear explicitly. This form is interesting for the reason that the steady state is classed as part of the transient solution. It appears in the light of a transient term for which the mode is zero. This, of course, still assumes a constant impressed force, i.e., a direct-current steady state.

In order to bring the form (914) in contact with the determinant form of solution, we note that:

$$Z^* = \frac{pD}{C} \qquad (915a)$$

where the indices are left off for convenience. This is the same as:

$$Z^* = \frac{p^n D}{p^{n-1}C} \qquad (915b)$$

where n is the number of meshes or the number of rows or columns in the modular determinant. If we define:

$$D^* = p^n D$$

and:

$$C^* = p^{n-1}C \qquad (916)$$

we recognize that D^* results from D by multiplying each element c_{ik} in D by p. C^* is then any principal minor of D^*. If we denote its elements by:

$$c_{ik}^* = pc_{ik} \qquad (917)$$

we see that:

$$c_{ik}^* = L_{ik}p^2 + R_{ik}p + S_{ik} \qquad (918)$$

which is what we would get if we wrote the equilibrium conditions for our network in terms of mesh charges instead of mesh-currents. Hence D^* is really the modular determinant of the system on the mesh charge basis. The impedance Z^* (when written as a function of $j\omega$ instead of p) is, therefore, the impedance which will give the steady-state mesh charge for a harmonic force function.

We merely mention these latter things in order to give Z^* a physical interpretation. They are not essential for a working knowledge of the form (914). Let us illustrate this form by means of the same example as above. Then we see that:

$$Z^* = Lp^2 + Rp + \frac{1}{C} \cdot \tag{919}$$

By (896) and (897) this is:

$$Z^* = L(p - p_1)(p - p_2). \tag{919a}$$

The equation:

$$Z^* = 0$$

contains no zero root. Hence, no steady state exists, which is correct. Now:

$$Z^{*\prime} = L[(p - p_1) + (p - p_2)] \tag{920}$$

so that:

$$\left.\begin{aligned} Z_1^{*\prime} &= L(p_1 - p_2) \\ Z_2^{*\prime} &= L(p_2 - p_1). \end{aligned}\right\} \tag{921}$$

Substituting this into (914), we get the same result as before.

The reader will observe that this form is quite a bit more convenient than the form (892), not only in appearance, but also in its application. The writer very much prefers the modified form (914).

On the basis of this modified form we may proceed to derive still another form for Heaviside's expansion formula which is a great deal more convenient to apply. This proceeds upon a suggestion which comes to us when we examine the expressions (919a), (920), and (921). These show that it is very convenient to express Z^* in terms of its roots, both from the standpoint of the differentiation as well as for the substitution of the roots. In any case we can always write:

$$Z^* = F(p) \cdot (p - p_1)(p - p_2) \ldots (p - p_m) \tag{922}$$

where:

$$p_1, p_2, \ldots p_m$$

are the m roots of

$$Z^* = 0.$$

The derivative of (922) is:

$$\frac{dZ^*}{dp} = F(p) \cdot \frac{d}{dp}[(p - p_1) \ldots (p - p_m)] + (p - p_1)(p - p_2) \ldots (p - p_m) \cdot \frac{dF}{dp} \cdot \qquad (923)$$

When we substitute any root of Z^* into this last expression, it is obvious that the second term will always vanish because it contains as factors the differences between p and any root. Hence only the first term need be considered. But this also becomes quite simple because:

$$\left[\frac{d}{dp}(p - p_1) \ldots (p - p_m)\right]_{p=p_\nu} = p_\nu(p_\nu - p_1)(p_\nu - p_2) \ldots (p_\nu - p_{\nu-1})(p_\nu - p_{\nu+1}) \ldots (p_\nu - p_m). \qquad (924)$$

All the rest of the terms vanish for the same reason as above. Hence:

$$Z_\nu^{*\prime} = p_\nu(p_\nu - p_1) \ldots (p_\nu - p_{\nu-1})(p_\nu - p_{\nu+1}) \ldots (p_\nu - p_m) \cdot F(p_\nu). \qquad (925)$$

This result together with (914) results in:

$$\dot{i}_{sk} = \sum_{\nu=0}^{m} \frac{Ee^{p_\nu t}}{p_\nu(p_\nu - p_1) \ldots (p_\nu - p_{\nu-1})(p_\nu - p_{\nu+1}) \ldots (p_\nu - p_m) \cdot F_{sk}(p_\nu)} \qquad (926)$$

which is another form in which Heaviside's formula is found. Note that the function F is the one which bears the indices relating to the location of the force and the mesh in which the current is calculated.

The formula (926) may be altered still more by introducing the determinantal notation. In fact there are an endless number of additional modifications. The form (926) is, however, about as convenient a one as may be obtained. It no longer contains any derivatives, but expresses the solution directly in terms of the modes and the function F.

As an illustration of the form (926), let us consider the pair of tuned coupled circuits of Fig. 114, but for zero resistance, and calculate the current in the second mesh. Then we have:

$$
\begin{aligned}
Z^* &= \frac{p(c_{11}c_{22} - c_{12}c_{21})}{-c_{12}} \\
&= \frac{p(c_{11} + c_{12})\,(c_{11} - c_{12})}{-c_{12}} \\
&= \frac{p}{-Mp}\left[(L + M)p + \frac{1}{Cp}\right]\left[(L - M)p + \frac{1}{Cp}\right] \\
&= -\frac{L^2 - M^2}{Mp^2}\left(p^2 + \frac{1}{C(L + M)}\right)\left(p^2 + \frac{1}{C(L - M)}\right) \\
&= -\frac{L^2 - M^2}{Mp^2}\,(p + j\omega_1)\,(p - j\omega_1)\,(p + j\omega_2)\,(p - j\omega_2). \qquad (927)
\end{aligned}
$$

Here it is obvious that:

$$F(p) = -\frac{L^2 - M^2}{Mp^2}. \qquad (928)$$

In substituting into (926) we note that the terms come in conjugate pairs. Hence, we shall consider only:

$$p = j\omega_1$$

and:

$$p = j\omega_2$$

and then take twice the real portion of the result. We have:

$$
\begin{aligned}
i &= 2\,\mathcal{R}_e\left[\frac{-Ee^{j\omega_1 t}\cdot M\omega_1^2}{2j\omega_1(\omega_1^2-\omega_2^2)\,(L^2-M^2)} + \frac{-Ee^{j\omega_2 t}\cdot M\omega_2^2}{2j\omega_2(\omega_2^2-\omega_1^2)\,(L^2-M^2)}\right] \\
&= \frac{EM}{(L^2-M^2)\,(\omega_2^2-\omega_1^2)}\,(\omega_1 \sin \omega_1 t-\omega_2 \sin \omega_2 t) \\
&= -\frac{EC}{2}\,(\omega_1 \sin \omega_1 t - \omega_2 \sin \omega_2 t) \qquad (929)
\end{aligned}
$$

which checks with the result (854). With the work thus highly organized, the formula does seem to be shorter than the ordinary method. However, with the same degree of organization, the usual method for obtaining the solution is just as brief. There may be some difference of opinion on this point, and the reader may, therefore, draw his own conclusions after he has become sufficiently familiar with the various methods.

The form (926), incidentally, makes it evident that the formula cannot be applied when coincident roots appear, because this would cause two or more factors in the denominator to be alike. Hence, the denominator would vanish for one of these roots, so that the entire expression would become infinite. Note also that a zero root of:

$$Z(p) = 0$$

leads to an infinite result because this means two zero roots for:

$$Z^*(p) = 0.$$

Thus, whenever the determinantal equation for the network contains a zero root, the formula does not apply. This would occur if we considered an inductance alone. Then:

$$Z(p) = Lp$$
$$Z^*(p) = Lp^2$$

so that:

$$Z^*(p) = Lp^2 = 0$$

contains two coincident roots at $p = 0$. However, practically, we never have inductance without also having some resistance, so that this exception is not serious.

To summarize the limitations of Heaviside's formula, we may state that it is not applicable for the cases where the initial conditions are arbitrary, or where Z^* is a constant, or where $Z^* = 0$ has coincident roots. Otherwise it may be applied in any of the forms given above. Of these (914) is the briefest analytically, and (926) is the most convenient for numerical application.

2. The Superposition Formula.[1] This formula is based upon the linearity of the network. It proceeds from the fact that the net behavior of the system at any instant is due to the linear superposition of all responses which have occurred up to that time, counting from some arbitrary starting point. Thus, if we should subject a network to an initial shock at $t = 0$, and then follow by subsequent applications of forces at specified times, the net behavior at any time would be a sum of the various results which have occurred up to that time, with due allowance for the time at which each individual response started. This idea is used in order to determine the behavior of a network when a force function of arbitrary form is impressed. So far we have considered only constant and harmonic forces. The object of the superposition formula is to give us a means for handling any kind of an impressed force.

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[1] For a thorough discussion see V. Bush, l.c., pp. 56 e.s.

Before giving the derivation of the formula proper, we must introduce a new kind of function which forms the starting point for this work. In the preceding paragraphs we have shown how the response of a network may be determined when a constant force is suddenly applied at some point. Suppose we consider the magnitude of this force to be unity, that it is applied in mesh #s, and that the response is desired in mesh #k. The function of time which represents this response. is denoted by:

$$A_{sk}(t) \tag{930}$$

and is called by Carson[1] the **indicial admittance** of mesh #s with respect to mesh #k. If the network is initially at rest, i.e., if the currents and charges are initially zero, then, due to the reciprocity theorem, the meshes s and k may be interchanged. The indicial admittance is thus seen to be the response of the network resulting from the sudden application of a unit force. It is the response per unit of applied force, with proper regard to the points of application and observation, as indicated by the indices. When this function is known, the response for any form of the applied force may be determined as follows.

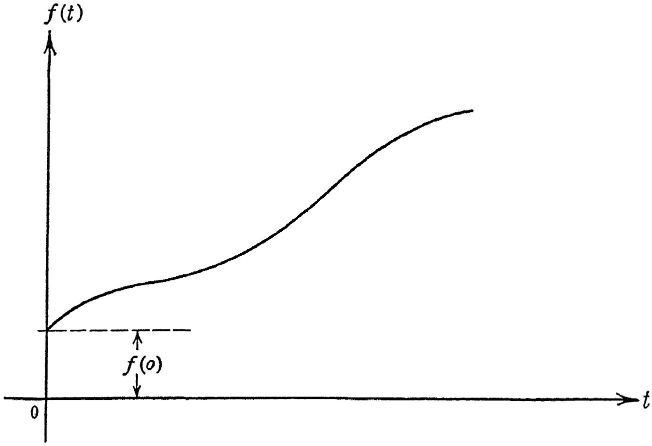

FIG. 128

Suppose we consider an applied force of the form illustrated in Fig. 128. Here $t = 0$ is the switching instant. At this instant the force has the value $f(0)$, i.e., it jumps from zero to $f(0)$ instantly.

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[1] A.I.E.E. Trans. Vol. 38, p. 345, 1919.

After this initial jump, it follows the continuous curve as indicated. It is a common concept in calculus that this continuous variation may be considered as made up of a succession of differential increments. In exaggerated form, this gives rise to the step-function illustrated in Fig. 128*a*. This function, made up of differential steps *df*, at differential time intervals *dt*, exactly replaces the original function in Fig. 128.

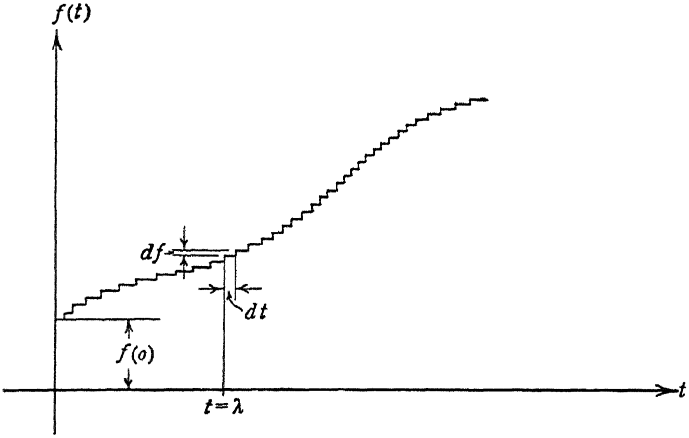

FIG. 128*a*

Now consider, for example, the differential force increment which takes place at a time $t = \lambda$, and is denoted by:

$$(df)_{t=\lambda}. \tag{931}$$

When this is suddenly impressed upon the network, the corresponding response is given by:

$$A(t - \lambda) \cdot (df)_{t=\lambda}. \tag{932}$$

The indices on A are left off for convenience. The reason that the argument of A is $t - \lambda$ and not just t is because this response does not come into existence until the time $t = \lambda$. That is, the expression (932) is valid only for $t \geqq \lambda$, and must reduce to $A(0)$ for $t = \lambda$. Since λ is an arbitrary point on the time scale, anywhere between zero and some specified time t, the expression (932) represents a typical differential response which may be initiated at any time between zero and t by the continuously varying force $f(t)$. Therefore, if we wish to know what the net response at the time t is, then we have merely to add up all the differential responses from zero to t, and add to this the initial response at the instant $t = 0$. This latter response is obviously given by:

$$f(0) \cdot A(t). \tag{933}$$

The sum of all the differential responses from zero to t becomes an integral of the expression (932) with respect to the variable λ, which is the time at which each differential response is initiated. In order to form this integral, we note that:

$$\left.\begin{aligned}(df)_{t=\lambda} &= \left(\frac{df}{dt}\right)_{t=\lambda} \cdot d\lambda \\ &= f'(\lambda) \cdot d\lambda\end{aligned}\right\} \tag{934}$$

where the prime indicates the differentiation with respect to time. Thus, we have:

$$i_{sk}(t) = f(0) \cdot A_{sk}(t) + \int_0^t A_{sk}(t - \lambda) f'(\lambda) d\lambda. \tag{935}$$

Note that the integration is with respect to the variable time λ at which each differential response starts. The result is a function of t, as it should be. This is the superposition formula. It expresses the response for any form of the applied force $f(t)$. It is not an explicit solution until the integral is evaluated. When the force function $f(t)$ is given in analytic form, the integral may be evaluated provided it is not too complicated. Otherwise the integration must either be performed graphically, i.e., in steps, or by means of a suitable form of integrating machine. When $f(t)$ is not given analytically, then the graphical or machine integration process is the only one possible. The real importance of the result (935) is that it expresses the response for any arbitrary applied force in terms of the response per unit of a constant applied force. Hence, the solution for any type of voltage is expressible in terms of the d.c. solution. This fact is very useful in network analysis.

The superposition formula (935), like Heaviside's formula, may be transformed into a number of equivalent forms, which may be better adapted to meet the requirements of particular problems. We shall now give the derivation of the more important of these. Suppose we introduce in the integral of (935) the change of variable indicated by:

$$t - \lambda = \delta. \tag{936}$$

Then:

$$d\lambda = -d\delta \tag{937}$$

where t is treated as a constant, which it is so far as the integration in (935) is concerned. The function under the integral sign then becomes:

$$-A(\delta)f'(t - \delta)d\delta \tag{938}$$

so that:

$$\begin{aligned}\int_0^t A(t - \lambda)f'(\lambda)d\lambda &= -\int_t^0 A(\delta)f'(t - \delta)d\delta \\ &= \int_0^t A(\delta)f'(t - \delta)d\delta\end{aligned}$$

and if we now write λ for δ again, we have instead of (935):

$$i_{sk} = f(0)A_{sk}(t) + \int_0^t f'(t - \lambda)A_{sk}(\lambda)d\lambda \tag{939}$$

which is one of the alternative forms.

Two additional forms are obtained from (935) and (939) respectively by carrying out an integration by parts. The well-known formula for this is:

$$\int u\,dv = uv - \int v\,du. \tag{940}$$

For (935), for example, we let:

$$\left.\begin{aligned} u &= A(t - \lambda) \\ dv &= f'(\lambda)d\lambda. \end{aligned}\right\} \tag{941}$$

Then since the differentiation and integration is with respect to λ we have:

$$\left.\begin{aligned} du &= -A'(t - \lambda)d\lambda \\ v &= f(\lambda). \end{aligned}\right\} \tag{942}$$

Hence, the integral in (935) becomes:

$$\begin{aligned}\int_0^t A(t - \lambda)f'(\lambda)\,d\lambda &= \Big[A(t - \lambda)f(\lambda)\Big]_0^t + \int_0^t f(\lambda)A'(t-\lambda)d\lambda \\ &= A(0)f(t) - A(t)f(0) + \int_0^t f(\lambda)A'(t-\lambda)d\lambda.\end{aligned} \tag{943}$$

Substituting this into (935), we have:

$$i_{sk} = A_{sk}(0)f(t) + \int_0^t f(\lambda)A_{sk}'(t - \lambda)\,d\lambda \tag{944}$$

which is the desired alternative.

In exactly the same manner we obtain from (939):

$$i_{sk} = A_{sk}(0)f(t) + \int_0^t A'(\lambda)f(t - \lambda)\,d\lambda. \tag{945}$$

Still other forms are obtained when we make use of the mathematical theory regarding the differentiation of functions under an integral sign. Briefly, the situation regarding this process is as follows. Suppose we consider the following function defined by an integral:

$$\phi(t) = \int_{h_1}^{h_2} f(t, \lambda)\,d\lambda \tag{946}$$

where f is a function of the two variables t and λ, and the limits of integration are functions of t. It will be seen that any one of the integrals involved in the above formulae have this general form. For this situation it is shown that:

$$\frac{d\phi}{dt} = \int_{h_1}^{h_2} \frac{\partial f}{\partial t}\,d\lambda + f(t, h_2)\cdot\frac{dh_2}{dt} + f(t, h_1)\cdot\frac{dh_1}{dt}\cdot \tag{947}$$

This result is very plausible because the differentiation with respect to t must affect the values at the limits, since these are functions of time. That is, ϕ must be looked upon as being a function of the variables t, h_1, and h_2, thus:

$$\phi = \psi(t, h_1, h_2)$$

whence:

$$\frac{d\phi}{dt} = \frac{\partial\psi}{\partial t}\frac{dt}{dt} + \frac{\partial\psi}{\partial h_1}\frac{dh_1}{dt} + \frac{\partial\psi}{\partial h_2}\frac{dh_2}{dt}\cdot$$

The form (947) is thus obtained.

With this in mind, consider:

$$\frac{d}{dt}\int_0^t A(t-\lambda)f(\lambda)\,d\lambda = \int_0^t A'(t-\lambda)f(\lambda)\,d\lambda + A(0)f(t). \tag{948}$$

Here we put:

$$\left.\begin{aligned} h_1 &= 0 \\ h_2 &= t. \end{aligned}\right\}$$

Similarly, we have:

$$\frac{d}{dt}\int_0^t f(t-\lambda)A(\lambda)\,d\lambda = \int_0^t f'(t-\lambda)A(\lambda)\,d\lambda + f(0)A(t). \quad (949)$$

Comparing (948) and (949) with (944) and (939) respectively, we see that:

$$i_{sk} = \frac{d}{dt}\int_0^t A_{sk}(t-\lambda)f(\lambda)\,d\lambda \quad (950)$$

and:

$$i_{sk} = \frac{d}{dt}\int_0^t f(t-\lambda)A_{sk}(\lambda)d\lambda \quad (951)$$

which are two new forms for the superposition formula. The chief difference between the last two and the preceding forms is the fact that the functions under the integral sign are not differentiated, the differentiation being done after the integration has been carried out. This may be advantageous since in some cases the process of differentiating makes a function more complicated, so that the subsequent integration becomes more difficult. It may, of course, be that the differentiation simplifies the function, in which case one of the other four forms would be preferable.

In general no statement can be made as to the preference of any one of the above forms over any other. This must be decided for each specific case. Such decisions require experience and familiarity with the various forms. Sometimes the best form is found only after a number of trials. We shall now consider several examples in order to illustrate some of the features discussed above.

It is interesting to show how, by means of the superposition formula, the solution for a harmonic force may be obtained when that for a constant force is given. Take, for example, a simple R, L circuit. For this we know that:

$$A(\lambda) = \frac{1 - e^{-\frac{R\lambda}{L}}}{R}. \quad (952)$$

The force function is given by:

$$f(t) = \mathcal{R}_e\,[Ee^{j\omega t}]. \quad (953)$$

Here we note that differentiating $A(\lambda)$ will actually simplify the resulting function under the integral. Hence, we shall use the form (945). We have:

$$A(0) = 0$$

$$A'(\lambda)\,d\lambda = \frac{e^{-\frac{R\lambda}{L}}}{L}\,d\lambda$$

so that:

$$\begin{aligned} i &= \int_0^t \Re_e\,[Ee^{j\omega(t-\lambda)}]\,\frac{e^{-\frac{R\lambda}{L}}}{L}\,d\lambda \\ &= \Re_e\left\{\frac{E}{L}e^{j\omega t}\int_0^t e^{-\left(\frac{R+jL\omega}{L}\right)\lambda}\,d\lambda\right\} \\ &= \Re^{e}\left\{\frac{-Ee^{j\omega t}}{R+jL\omega}\cdot e^{-\left(\frac{R+jL\omega}{L}\right)\lambda}\Big|_0^t\right\} \\ &= \Re_e\left[\frac{E}{R+jL\omega}\left(e^{j\omega t} - e^{-\frac{Rt}{L}}\right)\right] \end{aligned} \tag{954}$$

which we recognize as the correct result.

There is, of course, no advantage to be gained in arriving at the solution to this case by means of the superposition formula. The real utility of the latter is found in treating cases that are otherwise difficult to solve. However, we are giving these examples here merely for the purpose of illustrating the method. In this respect, consider the force function given by:

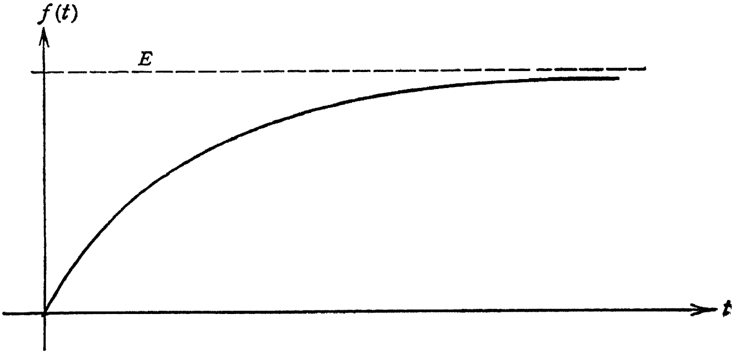

FIG. 129

$$f(t) = E(1 - e^{-\beta t}). \tag{955}$$

This is illustrated in Fig. 129. It is an exponentially increasing function which reaches the constant value E asymptotically. Suppose we consider this impressed upon an R, L, C circuit. We shall use the form (935) for this case. We have:

$$\left.\begin{aligned} f(0) &= 0 \\ f'(\lambda)\,d\lambda &= E\beta e^{-\beta\lambda}\,d\lambda. \end{aligned}\right\} \tag{956}$$

We also know that:

$$A(t) = \mathcal{R}_e\left[\frac{e^{(-\alpha+jg)t}}{jLg}\right] \tag{957}$$

where:

$$\alpha = \frac{R}{2L}$$

$$g = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

for the oscillatory case. Hence we get:

$$\begin{aligned} i &= \mathcal{R}_e \int_0^t \frac{e^{(-\alpha+jg)(t-\lambda)}}{jLg} E\beta e^{-\beta\lambda} d\lambda \\ &= \mathcal{R}_e \frac{E\beta e^{(-\alpha+jg)t}}{jLg} \int_0^t e^{(\alpha-\beta-jg)\lambda}\,d\lambda. \end{aligned} \tag{958}$$

This integral is easily evaluated. We get after taking the real portion:

$$i = \frac{EC\beta}{1 - RC\beta + LC\beta^2}\left[e^{-\beta t} - e^{-\alpha t}\left(\cos gt + \frac{\alpha - \beta}{g}\sin gt\right)\right]. \tag{959}$$

Usually:

$$\frac{\alpha - \beta}{g} \ll 1$$

so that:

$$i \cong \frac{EC\beta}{1 - RC\beta + LC\beta^2}\left(e^{-\beta t} - e^{-\alpha t}\cos gt\right). \tag{959a}$$

This result is particularly interesting when the decrement of the circuit equals the decrement of the force function, i.e., when:

$$\alpha = \beta. \tag{960}$$

Then (959a) becomes:

$$i = \frac{2\,ERC}{4\,L - R^2C} e^{-\alpha t}(1 - \cos gt). \tag{961}$$

Fig. 130 illustrates this case. The current oscillates and decays, but always remains positive. If $\alpha < \beta$, then the oscillation will go below the axis for a short time during each period.

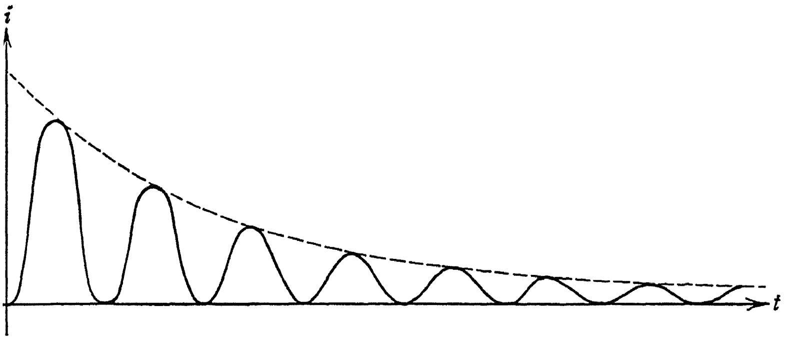

FIG. 130

Another interesting type of force function is that illustrated in Fig. 131. It is given analytically by the expression:

$$f(t) = Ete^{-\beta t}. \tag{962}$$

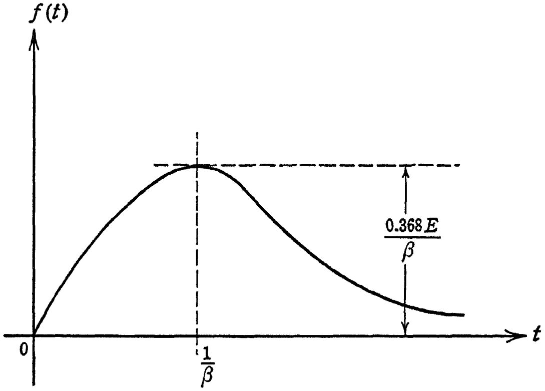

FIG. 131

We shall use the form (935) again. Here we have:

$$\left.\begin{aligned} f(0) &= 0 \\ f'(\lambda)\,d\lambda &= Ee^{-\beta\lambda}(1 - \beta\lambda)\,d\lambda. \end{aligned}\right\} \tag{963}$$

Consider this force impressed upon the R, L, C circuit for which the indicial admittance is given by (957). Then we get:

$$\begin{aligned} i &= \mathcal{R}_e \int_0^t \frac{e^{(-\alpha+jg)(t-\lambda)}}{jLg} Ee^{-\beta\lambda}(1 - \beta\lambda)\, d\lambda \\ &= \mathcal{R}_e \cdot \frac{Ee^{(-\alpha+jg)t}}{jLg} \int_0^t e^{(\alpha-\beta-jg)\lambda}(1 - \beta\lambda)\, d\lambda. \end{aligned} \tag{964}$$

After evaluating the integral, we have:

$$i = \mathcal{R} \cdot \frac{E}{jLg} \cdot \frac{[(\alpha-jg)\,(1-\beta t)+\beta^2 t]e^{-\beta t} + (-\alpha+jg)e^{(-\alpha+jg)t}}{(\alpha - \beta - jg)^2} \cdot \tag{965}$$

If we neglect the decrements as compared to the angular velocity, this becomes:

$$i \cong \frac{E}{Lg^2}[(1 - \beta t)e^{-\beta t} - e^{-\alpha t} \cos gt]. \tag{965a}$$

In particular for $\alpha = \beta$ we get:

$$i \cong \frac{Ee^{-\alpha t}}{Lg^2}[1 - \alpha t - \cos gt]. \tag{965b}$$

This last result is illustrated in Fig. 132.

All of these cases could have been treated by the classical method outlined in the preceding chapters, as well as by the superposition formula because the force functions were given analytically in the exponential form. However, when the analytic form of $f(t)$ is not an exponential, then the superposition method is better to use than an ordinary integration process. Of course, it may be quite difficult to carry out the integration in the superposition formula

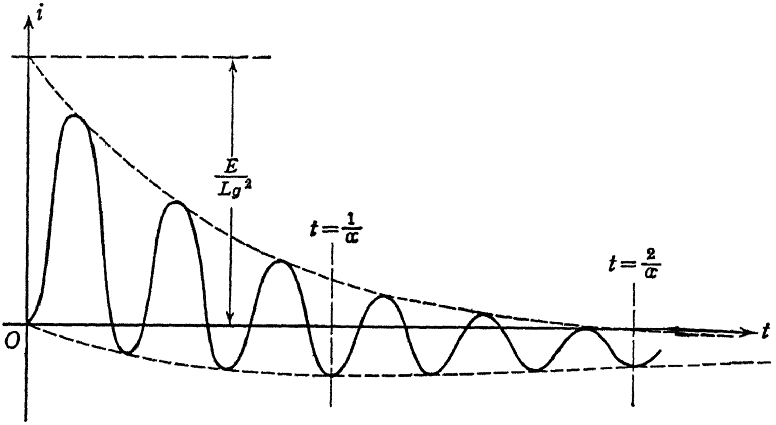

FIG. 132

for certain force functions, but the procedure is at least definitely outlined.

In conclusion we wish to mention the fact that the superposition formula is applicable to any linear system, not necessarily a lumped-constant network. It may, and often is, applied to distributed constant problems such as the uniform transmission line or cable. We shall, however, take up these points in more detail in a later chapter.

PROBLEMS TO CHAPTER IX

9–1. A two-mesh network is characterized by the following parameters:

$$\begin{aligned} R_{11} &= 200 & R_{22} &= 5 \\ L_{11} &= 0.1 & L_{22} &= 10^{-3} \\ L_{12} &= 10^{-3} & S_{22} &= 10^{9}. \end{aligned}$$

A constant force is suddenly applied in the first mesh. Assuming initial rest conditions, determine the currents in both meshes by means of Heaviside's formula and plot.

9–2. Assuming that the reaction of the second mesh in Prob. **9–1** upon the first is negligible, use the superposition formula for the determination of the current in the second mesh. Plot and compare with the solution of Prob. **9–1.**

9–3. Solve Prob. **9–1** omitting the resistance in the second mesh.

9–4. Solve Prob. **9–2** with the data of Prob. **9–3.**

9–5. Solve Prob. **9–1** omitting the inductance in the second mesh.

9–6. Solve Prob. **9–2** with the data of Prob. **9–5.**

9–7. Solve Prob. **9–1** omitting the capacitance in the second mesh.

9–8. Solve Prob. **9–2** with the data of Prob. **9–7.**

9–9. A two-mesh network is characterized by the following parameters:

$$\begin{aligned} R_{11} &= 500 & R_{22} &= 5 \\ S_{11} &= 10^{6} & L_{22} &= 10^{-3} \\ S_{12} &= 10^{6} & S_{22} &= 10^{9}. \end{aligned}$$

A constant force is suddenly applied in mesh #1. Assuming initial rest conditions, determine the currents in both meshes by Heaviside's formula and plot the result.

9–10. Assuming that the reaction of the second mesh in Prob. **9–9** upon the first is negligible, determine the current in the second mesh by the superposition formula and compare the result with that of Prob. **9–9.**

9–11. A two-mesh network is characterized by the following parameters:

$$\begin{aligned} R_{11} &= 200 & R_{22} &= 4 & L_{22} &= 10^{-3}. \\ L_{11} &= 0.1 & L_{22} &= 10^{-3} \\ S_{11} &= 10^{7} & S_{22} &= 10^{9} \end{aligned}$$

Assuming that the reaction of the second mesh upon the first is negligible, determine the current in the second mesh by the superposition formula and plot the result.

9–12. Check Prob. **9–11** by the classical method.

CHAPTER X

THE TREATMENT OF PERIODIC FORCE FUNCTIONS

1. The Complex Fourier Series. In this chapter we wish to discuss the method of attack which is used on network problems when periodic, but not harmonic, forces are involved. The basis of the method is due to the Fourier representation of a periodic function. The reader is expected to be somewhat familiar with this general subject, so that detailed proofs and discussions may be omitted. We shall, in the following, place the emphasis upon the *use* of Fourier series rather than upon their mathematical justification.

From the standpoint of utility, the Fourier series in the form of sines and cosines is extremely cumbersome. It is for this reason, more than any other, that many problems either remain unsolved or are referred to the laboratory for solution when an analytic method would yield a great deal more. This difficulty is very elegantly overcome by the complex form of series. The latter is extensively used by physicists, but quite generally unknown to engineers. Our first object, therefore, is to introduce the reader to this complex form.

We shall begin with the usual form of the Fourier series in terms of sines and cosines. This is:

$$f(t) = \alpha_0 + \sum_{\nu=1}^{\infty} \alpha_\nu \cos \nu\omega t + \sum_{\nu=1}^{\infty} \beta_\nu \sin \nu\omega t \tag{966}$$

where the coefficients are given by:

$$\left.\begin{aligned} \alpha_0 &= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\,dt \\ \alpha_\nu &= \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\cos \nu\omega t\,dt \\ \beta_\nu &= \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\sin \nu\omega t\,dt. \end{aligned}\right\} \tag{967}$$

Here ω is the fundamental angular velocity and ν is the index which specifies the order of the harmonic components. Time is the independent variable. This form is very inconvenient for network analysis. We saw in an early chapter how cumbersome the sine or cosine functions were when involved in the solution of differential equations. The Fourier series (966) involves an infinite sum of sines plus an infinite sum of cosines. If we attempted to use it as a force function in the solution of a network problem, the unwieldiness of the work would be altogether prohibitive. This is not the only objection to the above form. By (967) we see that three separate formulae are necessary for the determination of the coefficients. This also is a nuisance. The coefficient α_0, incidentally, is the constant component of the function $f(t)$.

It is possible to put (966) into the form of a cosine series or a sine series alone. Then each component has an individual phase angle in addition to a magnitude. This may easily be done as follows. If we use the rotating vector to represent a harmonic function, then the sine and cosine terms in (966) may be illustrated as in Fig. 133. The horizontal axis is chosen as the projection axis. The vector α_ν then represents the amplitude of a cosine term, because this starts with its positive maximum value at $t = 0$, which is the time for which the diagram is drawn. Similarly β_ν represents a sine term of νth order. These may be combined vectorially into γ_ν at an angle ϕ_ν as indicated. These vectors are to be thought of as about to rotate in the counterclockwise direction with an angular velocity of $\nu\omega$ radians per second, as indicated.

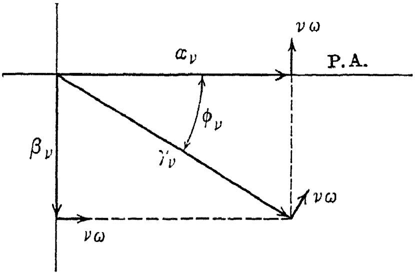

FIG. 133

From the figure it is clear that:

$$\left.\begin{aligned} \gamma_\nu &= \sqrt{\alpha_\nu^2 + \beta_\nu^2} \\ \phi_\nu &= \tan^{-1}\left(\frac{\beta_\nu}{\alpha_\nu}\right) \end{aligned}\right\} \qquad (968)$$

or inversely:

$$\left.\begin{aligned} \alpha_\nu &= \gamma_\nu \cos \phi_\nu \\ \beta_\nu &= \gamma_\nu \sin \phi_\nu. \end{aligned}\right\} \qquad (969)$$

Thus, we could write instead of (966):

$$f(t) = \alpha_0 + \sum_{\nu=1}^{\infty} \gamma_\nu \cos(\nu\omega t - \phi_\nu) \tag{966a}$$

or:

$$f(t) = \alpha_0 + \sum_{\nu=1}^{\infty} \gamma_\nu \sin\left(\nu\omega t + \frac{\pi}{2} - \phi_\nu\right). \tag{966b}$$

Although these forms require less space, they are in every respect as cumbersome to handle as (966). They require the formulae (967) plus (968) for the determination of the amplitudes and phases.

There is only one way out of this dilemma, and that is through the use of the exponential function. We know, for example, that:

$$\sin \nu\omega t = \frac{e^{j\nu\omega t} - e^{-j\nu\omega t}}{2j} \tag{970}$$

and:

$$\cos \nu\omega t = \frac{e^{j\nu\omega t} + e^{-j\nu\omega t}}{2}. \tag{970a}$$

If we substitute these expressions into (966) and arrange terms, we have:

$$f(t) = \alpha_0 + \sum_{\nu=1}^{\nu=\infty} \left\{ \frac{\alpha_\nu - j\beta_\nu}{2} e^{j\nu\omega t} + \frac{\alpha_\nu + j\beta_\nu}{2} e^{-j\nu\omega t} \right\}. \tag{971}$$

Now we note by (967) that:

$$\frac{\alpha_\nu - j\beta_\nu}{2} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} f(t) e^{-j\nu\omega t}\, dt \tag{972}$$

and:

$$\frac{\alpha_\nu + j\beta_\nu}{2} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} f(t) e^{j\nu\omega t}\, dt \tag{972a}$$

where we made use of the relation:

$$\cos \nu\omega t \pm j \sin \nu\omega t = e^{\pm j\nu\omega t}.$$

It is important to note that (972) and (972a) are conjugates, and that the only difference between the two integrals is the difference in the algebraic sign of the index ν. This suggests that we introduce the new notation:

$$\left.\begin{aligned} a_\nu &= \frac{\alpha_\nu - j\beta_\nu}{2} \\ a_{-\nu} &= \frac{\alpha_\nu + j\beta_\nu}{2}. \end{aligned}\right\} \tag{973}$$

Then obviously:

$$a_\nu = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} f(t)e^{-j\nu\omega t}\,dt. \tag{974}$$

Note that the single formula (974) now replaces the two formulae (972) and (972*a*), since the latter is obtained from (974) by merely changing the sign of ν. Also, note that for $\nu = 0$, (974) becomes:

$$a_0 = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\,dt$$

which is identical with the first formula (967). Hence, (974) also gives the constant component of the series, i.e.:

$$a_0 = \alpha_0. \tag{975}$$

In terms of these results we may now write the expression (971):

$$f(t) = \sum_{\nu=-\infty}^{\nu=\infty} a_\nu e^{j\nu\omega t} \tag{976}$$

in which the two terms in the sum of (971), as well as the constant term α_0, are included by virtue of the fact that the index ν now takes on all negative integer values, as well as all positive integer values, inclusive of the value zero. Thus, the result (976) identically replaces (966), when the coefficients are determined by (974). This is called the **complex Fourier series.** Note how compact it is as compared to the clumsy form (966). Note also that only the one formula (974) is necessary for the determination of the coefficients as compared to the three formulae (967).

The complex form has so many advantages over the sine and cosine form that it is hard to appreciate them all at once. Processes of differentiation and integration with the form (976) are as simple as they are for a single exponential function. Another important point from the standpoint of network theory is the fact that no harmonic phase angles appear explicitly in the complex form; yet they are contained in it, namely in the complex character of the coefficients. Due to the fact that they are thus hidden from view, they cannot bother us when they are not wanted, as, for instance, during an integration process. After the integration is complete, the phases of the various harmonic components are immediately apparent by inspection of the resulting coefficients. This will be appreciated more fully later on.

Note also that although the separate terms in (976) are complex, the sum is not complex, but a real function. This is due to the fact that the terms come in conjugate pairs, each pair being given by the positive and negative values of the index ν for a certain integer. Such a pair of conjugate terms *together* represent *one harmonic component.* Thus, consider, for example, the pair of conjugate terms:

$$a_\nu e^{j\nu\omega t} + a_{-\nu}e^{-j\nu\omega t}. \tag{977}$$

Each term is a vector which rotates with an angular velocity equal to ω times the order of the harmonic. This pair of rotating vectors is illustrated in Fig. 134. Due to the fact that they are conjugates, their vector sum is real. This real value is the same as the projection of the vector γ_ν upon the real axis in Fig. 133. The magnitude of the vectors in Fig. 134 is half of the magnitude of γ_ν. The angle which the pair of vectors make with the horizontal is ϕ_ν, and is the same in magnitude as in the Fig. 133 or as given by (968). Thus, we can see by inspection of these two figures that the equations (973) are correct.

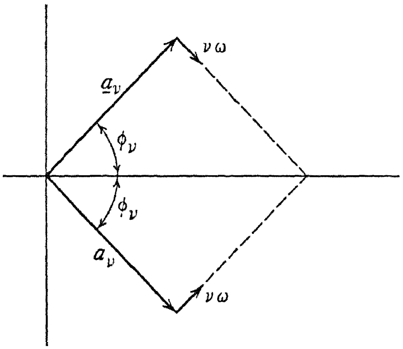

Fig. 134

The essence of the complex form (976) is, that each harmonic component is represented by a pair of oppositely rotating vectors. This is an old method for representing a harmonic function. The same device is made use of in the theory of the single phase induction motor where the harmonically pulsating field is replaced by two oppositely rotating fields. The phase angle of a harmonic component is given by the angle of the corresponding complex coefficient as illustrated in Fig. 134. In particular when the phase angle of a_ν is minus ninety degrees, then the conjugate pair of vectors represents a sine function. When the angle of a_ν is zero, then the pair defines a cosine function. The actual amplitude of any harmonic component is always twice the magnitude of the corresponding a_ν.

These matters will become clearer by means of some illustrative examples. Consider the square function given in Fig. 135. It is assumed to continue in the same fashion in both directions. The origin for the time is arbitrarily chosen as indicated. The positive

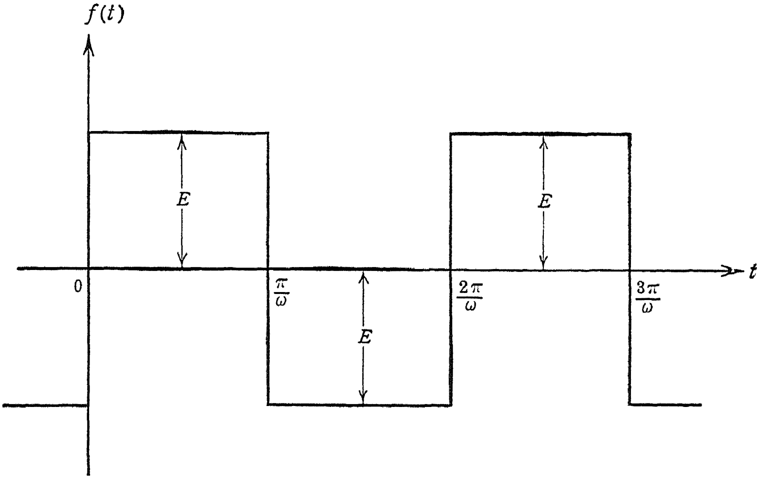

FIG. 135

and negative amplitudes are equal and denoted by E. By (974) we then have:

$$a_\nu = \frac{E\omega}{2\pi}\left[\int_0^{\frac{\pi}{\omega}} e^{-j\nu\omega t}\,dt - \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-j\nu\omega t}\,dt\right] \tag{978}$$

which is equivalent to:

$$a_\nu = \frac{E\omega}{2\pi}(1 - e^{-j\nu\pi})\int_0^{\frac{\pi}{\omega}} e^{-j\nu\omega t}\,dt. \tag{978a}$$

Integrating and substituting limits, we have:

$$a_\nu = \frac{E}{2\pi j\nu}(1 - e^{-j\nu\pi})^2. \tag{979}$$

The factor $(1 - e^{-j\nu\pi})^2$ is rather interesting. For ν equal to zero or any even integer, the factor vanishes, which indicates that the function of Fig. 135 contains no constant component and no even harmonics. For any odd integer value of ν, the factor is always equal to *four*. Hence, for the odd harmonic components we get:

$$a_\nu = \frac{2E}{j\pi\nu}. \tag{979a}$$

The complete Fourier series is, therefore:

$$f(t) = \sum_{\nu=-\infty}^{\infty} \frac{2E}{j\nu\pi} e^{j\nu\omega t}. \tag{980}$$

Since the angle of a_ν as given by (979a) is minus ninety degrees, the series (980) must be a sine series. If we group positive and negative terms in (980), we find:

$$f(t) = \sum_{\nu=1}^{\infty} \frac{4E}{\pi\nu} \sin \nu\omega t \tag{980a}$$

which we recognize as the Fourier series of the square wave with the time origin as indicated in Fig. 135. However, for purposes of analytic manipulation the complex form (980) is much to be preferred, even over the simple sine series.

If we had taken the time-origin at the center of the positive half-cycle instead of at the beginning of it, then the result would have been a cosine instead of a sine series. If the time origin is taken at some intermediate point, then the series is equivalent to a sine plus a cosine series, or either a sine or cosine series with phase angles. Suppose, for example, that the time origin in Fig. 135 is shifted to a point $t = \frac{\theta}{\omega}$. Then the result is immediately obtained from (980) by the change of variable:

$$t = t' - \frac{\theta}{\omega} \tag{981}$$

where t' is the time measured from the new origin. Substituting (981) into (980), we have:

$$f(t') = \sum_{\nu=-\infty}^{\infty} \frac{2E}{j\nu\pi} e^{-j\nu\theta} \cdot e^{j\nu\omega t'}. \tag{982}$$

The harmonic coefficients are obviously:

$$a_\nu = \frac{2E}{j\nu\pi} \cdot e^{-j\nu\theta} \tag{983}$$

with the same restriction as to odd integer values of ν as before. Thus, the time origin is easily shifted to any point, as may be required by special switching conditions.

As another example consider the triangular function shown in Fig. 136. This, it will be noted, is the integral of the periodic function given in Fig. 135 with due regard to its periodicity. By the last remark we mean that the square wave of Fig. 135 must be so integrated as to yield a function which contains no constant component. Here the function is defined as follows:

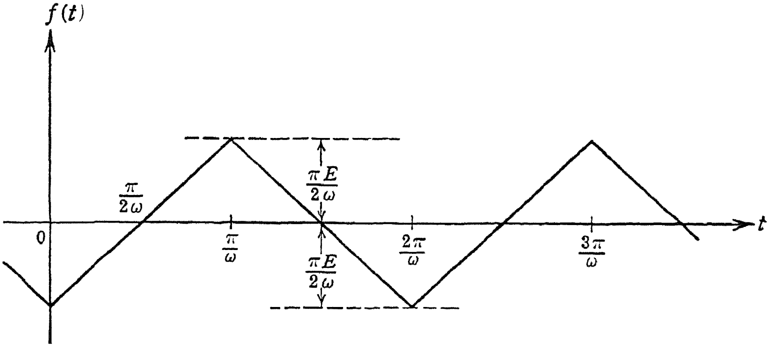

FIG. 136

$$\left.\begin{aligned} f(t) &= -\frac{E\pi}{2\omega}\left(1 - \frac{2\,\omega t}{\pi}\right) \quad \text{for:} \quad 0 < t < \frac{\pi}{\omega} \\ f(t) &= \frac{E\pi}{2\omega}\left(3 - \frac{2\,\omega t}{\pi}\right) \quad \text{for:} \quad \frac{\pi}{\omega} < t < \frac{2\,\pi}{\omega}\cdot \end{aligned}\right\} \tag{984}$$

By (974) we, therefore, have:

$$a_\nu = -\frac{E}{4}\left[\int_0^{\frac{\pi}{\omega}}\left(1 - \frac{2\,\omega t}{\pi}\right)e^{-j\nu\omega t}\,dt - \int_{\frac{\pi}{\omega}}^{\frac{2\,\pi}{\omega}}\left(3 - \frac{2\,\omega t}{\pi}\right)e^{-j\nu\omega t}\,dt\right]. \tag{985}$$

After carrying out the integration we have:

$$a_\nu = \frac{-E}{\pi\nu^2\omega}(1 - e^{-j\nu\pi}). \tag{986}$$

On account of the factor $(1 - e^{-j\nu\pi})$, this expression vanishes for even values of ν and also for ν equal to zero. For odd integer values of ν we have:

$$a_\nu = -\frac{2\,E}{\pi\nu^2\omega}\cdot \tag{987}$$

Hence, the Fourier series for the function given by Fig. 136 becomes:

$$f(t) = \sum_{\nu=-\infty}^{\infty} \frac{-2\,E}{\pi\nu^2\omega} \cdot e^{j\nu\omega t}. \tag{988}$$

Note that this is exactly what we get by integrating the series (980) term by term, which agrees with the fact that the function of Fig. 136 is the integral of the function of Fig. 135.

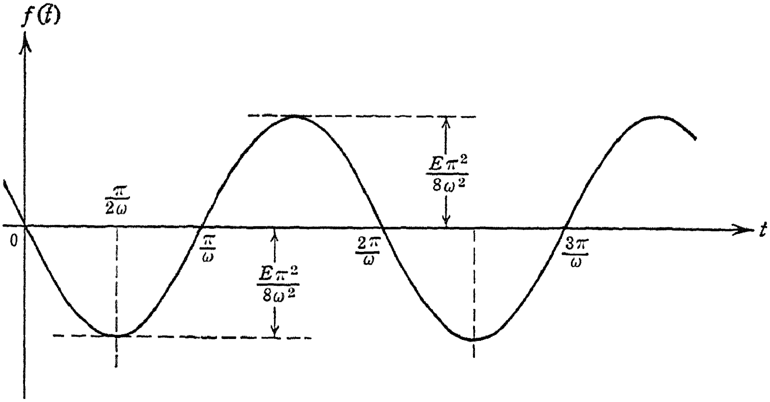

Fig. 137

Let us continue in this same way. The integral of the periodic function of Fig. 136 is given in Fig. 137. It consists of a succession of parabolae whose axes coincide with:

$$t = \frac{\pi}{2\,\omega}$$

$$t = \frac{3\,\pi}{2\,\omega}$$

etc.

The height of the apex of each parabolic section is equal to the area under the triangle of Fig. 136 for any quarter of a period. This may easily be verified by integrating the curve of Fig. 136.

The function of Fig. 137 is thus found to be defined as follows:

$$\left.\begin{aligned} f(t) &= -\frac{E}{2}t\left(\frac{\pi}{\omega} - t\right) \quad \text{for:}\quad 0 < t < \frac{\pi}{\omega} \\ f(t) &= -\frac{E}{2}\left(\frac{\pi}{\omega} - t\right)\left(\frac{2\,\pi}{\omega} - t\right) \quad \text{for:}\quad \frac{\pi}{\omega} < t < \frac{2\,\pi}{\omega}. \end{aligned}\right\} \tag{989}$$

Hence by (974) we have:

$$a_\nu = -\frac{E\omega}{4\pi}\left[\int_0^{\frac{\pi}{\omega}}\left(\frac{\pi}{\omega} - t\right)te^{-j\nu\omega t}\,dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}}\left(\frac{\pi}{\omega} - t\right)\left(\frac{2\pi}{\omega} - t\right)e^{-j\nu\omega t}\,dt\right]. \tag{990}$$

Carrying out the integration we find:

$$a_\nu = \frac{jE}{\pi\nu^3\omega^2}(1 - e^{-j\nu\pi}). \tag{991}$$

Here again the coefficients vanish for even integer values of ν and also for $\nu = 0$. For odd integer values we have:

$$a_\nu = \frac{2jE}{\pi\nu^3\omega^2} \tag{992}$$

so that the Fourier series for the function of Fig. 137 becomes:

$$f(t) = \sum_{\nu=-\infty}^{\infty} \frac{2jE}{\pi\nu^3\omega^2}e^{j\nu\omega t}. \tag{993}$$

This we see is exactly the term by term integral of (988), which is the series for the triangular wave of Fig. 136.

2. Some Remarks About Convergence. Our object in working out these three cases, which are successive integrals of each other, has not been merely to illustrate the application of the formula (974) for the determination of the coefficients of the complex series. We wish to call attention to a very interesting point in connection with these functions and their respective Fourier series. Consider in order, the square wave of Fig. 135, the triangular wave of Fig. 136, and the parabolic wave of Fig. 137. The first of these is quite unlike a sine wave; the second is somewhat nearer to it in shape; while the third is already so much like a sine wave in shape that it is hard to tell from the figure that it is not.

With this in mind consider the corresponding Fourier series for these cases as given by (980), (988), and (993) respectively. In (980) we see that the magnitudes of the harmonic components are inversely proportional to the index ν. That is, the third harmonic is one-third as large as the fundamental; the fifth is one-fifth of the fundamental; etc. We say that this series converges as *one over* ν. This convergence for the square wave is not very rapid, i.e., the harmonics are relatively quite large. For the triangular wave whose series is given by (988), however, the amplitudes of the harmonic components are inversely proportional to ν^2. Here the third harmonic is one-ninth as large as the fundamental; the fifth is one-twenty-fifth as large as the fundamental; etc. This series converges as *one over* ν^2. Finally, for the series (993), which represents the parabolic wave, we see that the third harmonic is only one-twenty-seventh of the fundamental; the fifth is one-one-hundred-and-twenty-fifth; etc. The series converges as *one over* ν^3.

We may correlate these results in the following way, in terms of the discontinuities of the function and its derivatives. Note, for example, that the function representing the square wave has discontinuities at the points: $t = 0, \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \ldots$ etc. Whenever the function has discontinuities, then the resulting series will converge no better than one over ν. When the function itself is continuous, but its first derivative has discontinuities, then the resulting series will converge no better than one over ν^2. This is the case with the triangular wave. Its first derivative has discontinuities at the points: $t = 0, \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \ldots$ etc. When the function and its first derivative are continuous, but its second derivative has discontinuities, then the resulting series will converge no better than one over ν^3. The parabolic wave has discontinuities in its second derivative at the points: $t = 0, \frac{\pi}{\omega}, \frac{2\pi}{\omega}$, . . . etc., i.e., at the points where the parabolic sections are pieced together.

It may be demonstrated that when the periodic function itself is continuous, but the nth derivative has discontinuities, then the resulting series will converge no better than one over ν^{n+1}. This is a very useful theorem to remember when expanding a function in a Fourier series. By means of it we often know beforehand how well or how poorly the series will converge. It also serves as a partial check on the numerical work in some problems.

Note that when a function and its derivatives are continuous only up to the nth, then higher derivatives do not exist, i.e., they cease to have any meaning. For example, it is meaningless to speak of the derivative of the square wave, or the second derivative of the triangular wave, etc. This fact is reflected in the form of the corresponding Fourier series. If we differentiate the series (980) for the square wave, we get:

$$\sum_{\nu=-\infty}^{\infty} \frac{2\,E\omega}{\pi}\,e^{j\nu\omega t}.$$

Here the coefficients do not contain ν at all. Hence, this series no longer converges. The series (988) for the triangular wave may be differentiated once. It will then yield the square wave. It may not be differentiated again for the same reason that the square wave could not be differentiated. The series (993) for the parabolic wave may be differentiated term by term twice. Differentiating makes the convergence poorer just as integrating makes it better. When the series has been differentiated until it converges as one over ν, then it can be differentiated no more.

These facts are quite useful in connection with network problems. For example, in an inductive circuit the current cannot be discontinuous, but its first derivative will have discontinuities when the force function has such discontinuities. Hence, in such a case the Fourier series for current can converge no better than one over ν^2. When the circuit is assumed to contain only resistance and capacitance, then the current series can converge no better than one over ν when the force function contains discontinuities. The series for charge will always converge as one over ν better than the corresponding current series. These and related facts aid in checking results in connection with network problems involving Fourier series.

3. Effective Values and Power Products. The determination of root-mean-square or effective values of periodic functions is a common problem. We shall show how this may be done with the complex series. Suppose we have a current given by:

$$i = \sum_{\nu=-\infty}^{\infty} a_\nu e^{j\nu\omega t} \tag{994}$$

and we wish to find its effective value. We know that:

$$i_{\text{eff}}^2 = \frac{\omega}{2\,\pi}\int_0^{\frac{2\,\pi}{\omega}} i^2\,dt. \tag{995}$$

In order to substitute into this formula we must square the series (994). This is merely the series multiplied by itself, thus:

$$i^2 = \sum_{\nu=-\infty}^{\infty} a_\nu e^{j\nu\omega t} \cdot \sum_{\mu=-\infty}^{\infty} a_\mu e^{j\mu\omega t}$$

$$= \sum_{\substack{\nu=-\infty \\ \mu=-\infty}}^{\infty} a_\mu a_\nu e^{j(\nu+\mu)\omega t} \tag{996}$$

where two different letters are used as indices to avoid confusion. Substituting (996) into (995), we have:

$$i_{\text{eff}}^2 = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \sum_{\substack{\nu=-\infty \\ \mu=-\infty}}^{\infty} a_\mu a_\nu e^{j(\nu+\mu)\omega t}\, dt. \tag{997}$$

This integration may be carried out term by term, so that we have:

$$i_{\text{eff}}^2 = \frac{\omega}{2\pi} \sum_{\substack{\nu=-\infty \\ \mu=-\infty}}^{\infty} a_\mu a_\nu \int_0^{\frac{2\pi}{\omega}} e^{j(\nu+\mu)\omega t}\, dt. \tag{998}$$

Now we know that:

$$\int_0^{\frac{2\pi}{\omega}} e^{jm\omega t}\, dt = \frac{e^{jm\omega t}}{jm\omega}\Bigg|_0^{\frac{2\pi}{\omega}}$$

$$= \frac{e^{jm2\pi} - 1}{jm\omega} = 0 \tag{999}$$

where m is any integer. Hence, all the integrals in (998) vanish except those for which:

$$\mu = -\nu. \tag{1000}$$

Thus, we have:

$$i_{\text{eff}}^2 = \frac{\omega}{2\pi} \sum_{\nu=-\infty}^{\infty} a_\nu a_{-\nu} \int_0^{\frac{2\pi}{\omega}} dt$$

$$= \sum_{\nu=-\infty}^{\infty} a_\nu a_{-\nu} \tag{1001}$$

which is the desired result. We may put this in another form if we recognize that the quantity under the summation sign in (1001) is the square of the magnitude of a_ν. Also, the summation over negative integers is the same as that over positive integers, so that we have:

$$i_{\text{eff}}^2 = 2\sum_{\nu=1}^{\infty} |a_\nu|^2. \tag{1001a}$$

Since the magnitude of a_ν is half the amplitude γ_ν of the corresponding harmonic, we may also write:

$$i_{\text{eff}}^2 = \tfrac{1}{2}\sum_{\nu=1}^{\infty} \gamma_\nu^2 \tag{1001b}$$

which is the commoner form for this result.

Note how simple it is to square the complex series and to integrate the result. The reader should carry through the same problem with the sine and cosine form (966) to appreciate the convenience of the complex form.

Another typical problem is that of finding the power absorbed by some device when the impressed voltage and the current are both given by Fourier series. Suppose we have:

$$e = \sum_{\nu=-\infty}^{\infty} a_\nu e^{j\nu\omega t} \tag{1002}$$

and:

$$i = \sum_{\mu=-\infty}^{\infty} b_\mu e^{j\mu\omega t}. \tag{1003}$$

The power is then given by:

$$P = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} ei\,dt. \tag{1004}$$

Substituting and interchanging the order of summation and integration, we get:

$$P = \frac{\omega}{2\pi}\sum_{\substack{\nu=-\infty \\ \mu=-\infty}}^{\infty} a_\nu b_\mu \int_0^{\frac{2\pi}{\omega}} e^{j(\nu+\mu)\omega t}\,dt. \tag{1005}$$

This is exactly the same integral that we had to evaluate before. Hence, the result is:

$$P = \sum_{\nu=-\infty}^{\infty} a_\nu b_{-\nu}. \tag{1006}$$

This may be converted into a sum over positive integers only if we note that:

$$a_\nu b_{-\nu} + a_{-\nu} b_\nu = 2\,\mathcal{R}e[a_\nu b_{-\nu}]. \tag{1007}$$

Then:

$$P = 2\,\mathcal{R}_e \sum_{\nu=1}^{\infty} a_\nu b_{-\nu}. \tag{1006a}$$

Note that the relative phase positions between the voltage and current harmonics are taken care of by the complex character of the coefficients. No phase angles appear explicitly, and consequently they cannot cause bother and confusion while carrying out the integration. This result is interesting in that it shows that the net power is due only to the power products of the corresponding harmonics. That is, no power is contributed by the fundamental of the voltage with the third harmonic of the current, or any other such cross-combination. If a harmonic voltage were impressed upon a circuit, and the resulting current contained harmonics, (due, for example, to a non-linear circuit element such as an iron core inductance) then the power absorbed by that circuit would be simply the power product of the voltage and the fundamental current component. The current harmonics could contribute nothing to the net power.

4. Problems Involving Periodic Force Functions. When the force function in a network problem is given in the form of a complex Fourier series, then the solution is practically as simple to obtain as it is for a harmonic function. We shall discuss here only the steady-state solution. The transient solution has the same form no matter what the driving force is. Suppose we consider an R, L, C circuit with a non-harmonic impressed force given by:

$$e = \sum_{\nu=-\infty}^{\infty} a_\nu e^{j\nu\omega t}. \tag{1008}$$

Then the equilibrium condition becomes:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = \sum_{\nu=-\infty}^{\infty} a_\nu e^{j\nu\omega t}. \tag{1009}$$

The heuristic method now suggests that the steady-state current be given by a Fourier series also, i.e., that we should assume:

$$i = \sum_{\nu=-\infty}^{\infty} b_\nu e^{j\nu\omega t}. \tag{1010}$$

In order to substitute this assumption into (1009), we must form the derivative and the integral of this series. This may be carried out term by term, thus:

$$L\frac{di}{dt} = L\sum_{\nu=-\infty}^{\infty} j\nu\omega b_\nu e^{j\nu\omega t} \tag{1011}$$

and:

$$\frac{1}{C}\int i\,dt = \frac{1}{C}\sum_{\nu=-\infty}^{\infty} \frac{b_\nu}{j\nu\omega} e^{j\nu\omega t}. \tag{1012}$$

Notice that the exponential series is as simple to manipulate as a single harmonic function in complex form. The substitution of (1010) into (1009), therefore, becomes:

$$\sum_{\nu=-\infty}^{\infty}\left\{R + j\left(L\nu\omega - \frac{1}{C\nu\omega}\right)\right\} b_\nu e^{j\nu\omega t} = \sum_{\nu=-\infty}^{\infty} a_\nu e^{j\nu\omega t} \tag{1013}$$

where the terms on the left-hand side have been placed under a single summation sign and grouped in a convenient manner. This equation is to be regarded as a condition which must be satisfied for all values of time in order to make the assumption (1010) valid. It is not hard to satisfy this condition. In fact it is quite obvious that it will be satisfied if we make the coefficients of the exponential functions in the two sums equal to each other. That is, if:

$$\left\{R + j\left(L\nu\omega - \frac{1}{C\nu\omega}\right)\right\} b_\nu = a_\nu \tag{1014}$$

then (1013) becomes an identity. By (1014) we have:

$$b_\nu = \frac{a_\nu}{R + j\left(L\nu\omega - \frac{1}{C\nu\omega}\right)} \tag{1015}$$

which expresses the Fourier coefficients for the current in terms of those for the voltage. The forms (1010) and (1015), therefore, constitute the steady-state solution to this problem.

In order to condense the notation somewhat, we may define:

$$X_\nu = L\nu\omega - \frac{1}{C\nu\omega} \tag{1016}$$

as the reactance for the νth harmonic. Then:

$$Z_\nu = R + jX_\nu \tag{1017}$$

is the impedance to the νth harmonic. The expression (1015) then reads:

$$b_\nu = \frac{a_\nu}{Z_\nu} \tag{1018}$$

which states that any current coefficient is equal to the corresponding voltage coefficient divided by the complex impedance for that harmonic as indicated by the index ν. Thus, the result merely substantiates again the superposition principle in linear networks; namely that when several voltages are simultaneously impressed, the resulting current is a linear sum of the current responses for the individual voltages. Here the latter are in the form of Fourier components; but the superposition principle which holds generally, is again evident.

Note that the reactance (1016) has the usual form for an R, L, C circuit except that here $\nu\omega$ appears where simply ω appears in the solution for a harmonic force function. Thus, the inductive reactance increases with the order of the harmonic, whereas the capacitive reactance decreases. This means that for a certain region of harmonics, the reactance will be relatively small as compared to harmonics which are some distance away from this region. The response is, therefore, selective as to harmonics just as it is selective as to frequency for this circuit. The analysis on the basis of harmonics is the same as it is for frequency except that ν is a discontinuous variable whereas ω is continuous.

In order to illustrate this let us consider the following numerical example with:

$$\left.\begin{aligned} L &= 4 \cdot 10^{-2} \\ C &= 10^{-6} \\ R &= 50 \end{aligned}\right\}$$

We will suppose that the force function is the square wave given by (980), i.e., that:

$$e = \sum_{\nu=-\infty}^{\infty} \frac{2\,E}{j\nu\pi} e^{j\nu\omega t} \tag{1019}$$

with:

$$\frac{2\,E}{\pi} = 1000.$$

We find by (1016) and (1017):

$$\begin{aligned}
X_1 &= -960 & |Z_1| &= 962\\
X_3 &= -213 & |Z_3| &= 219\\
X_5 &= 0 & |Z_5| &= 50\\
X_7 &= 137 & |Z_7| &= 146\\
X_9 &= 249 & |Z_9| &= 254\\
X_{11} &= 349 & |Z_{11}| &= 353\\
X_{13} &= 443 & |Z_{13}| &= 446\\
X_{15} &= 533 & |Z_{15}| &= 535.
\end{aligned}$$

We have determined only the magnitudes of the impedances since we are primarily interested in the magnitudes of the harmonics. According to (1018), these are given by:

$$|b_\nu| = \frac{1000}{\nu |Z_\nu|}\cdot \tag{1020}$$

We find:

$$\left.\begin{aligned}
|b_1| &= 1.04 & |b_3| &= 1.52\\
|b_5| &= 4.00 & |b_7| &= 0.98\\
|b_9| &= 0.437 & |b_{11}| &= 0.258\\
|b_{13}| &= 0.172 & |b_{15}| &= 0.125.
\end{aligned}\right\} \tag{1021}$$

The actual magnitudes of the harmonics are double these values. However, we are more interested in the relative magnitudes. In Fig. 138 is plotted the distribution of harmonics given by (1021). The relative magnitudes are indicated by vertical lines. The fundamental corresponds to an angular frequency of 1000 radians per second.

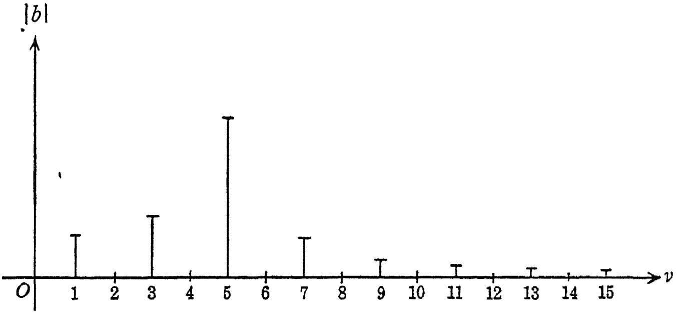

FIG. 138

We see that the fifth harmonic predominates considerably over the others. The circuit parameters were chosen so that resonance would occur at this frequency. This principle of harmonic selection is a very common method used for frequency multiplication. The method of frequency multiplication by means of the ordinary arc is based upon this principle. Here an almost square wave of voltage is produced at the terminals of the arc by an appropriate primary circuit. The arc voltage is impressed upon a secondary circuit which is tuned to select a higher frequency as in the above example. After this frequency has been selected it may be "refined" by means of a further selective system based upon the principles outlined at the close of Chapter VIII. The multiplication may be done in several successive stages instead of one. When higher multiples are desired, it is necessary from the standpoint of efficiency and quality to use more than one stage, with some form of refining circuit interposed. Vacuum tubes are also used in conjunction with this same idea. Basically, however, the same principle of harmonic selection, as that illustrated in the above example, is used in all these schemes. When the Fourier analysis of the force function is known, the solution to such problems is very simple.

It is interesting to put the solution for the R, L, C circuit with a square voltage wave impressed into its analytic form. By (979a) the coefficients (1015) for the current become:

$$b_\nu = \frac{2E}{j\pi\nu(R + jX_\nu)} \tag{1022}$$

where the abbreviation (1016) for the reactance is used. This expression together with (1010) give the desired current solution.

In many cases it may be necessary to derive the expression for the power absorbed by the circuit under these conditions. This is easily done by determining the squared effective current and multiplying this by the resistance. By (1001) we have in the above case:

$$i_{\text{eff}}^2 = \sum_{\nu=-\infty}^{\infty} b_\nu b_{-\nu}. \tag{1023}$$

Since only odd harmonics are present in our present problem, we have by extending the summation over positive integers only:

$$i_{\text{eff}}^2 = 2 \sum_{\nu=1,3,5\ldots}^{\infty} b_\nu b_{-\nu}. \tag{1023a}$$

By (1022) we have:

$$b_\nu b_{-\nu} = \frac{4\,E^2}{\pi^2\nu^2(R^2 + X_\nu^2)} \tag{1024}$$

so that the absorbed power becomes:

$$P = \sum_{\nu=1,3,5\ldots}^{\infty} \frac{4\,RE^2}{\pi^2\nu^2(R^2 + X_\nu^2)}. \tag{1025}$$

Note how simple the complex forms make these manipulations. This result for the absorbed power is not yet in the most convenient form for purposes of calculation because of the summation. Ordinarily the sum will converge quite rapidly, but it is still better to have the result in a closed form. This may be done by carrying out the above summation. We shall show how this may be done in a later section of this chapter.

We wish to take up now the same R, L, C circuit, but with another type of periodic force function. This is illustrated in Fig. 139. It consists essentially of a series of positive and negative voltage pulses which have a fundamental period of $\frac{2\,\pi}{\omega}$ seconds. The time origin is so chosen as to make each pulse come in the center of a half-period. This makes the expansion in a Fourier series simpler. The duration of a pulse is denoted by τ. This sort of a force function is very common in communication work. In many types of problems we find that an oscillatory circuit is energized by means of a succession of pulses such as the above. In representing the individual pulses by rectangles we are, of

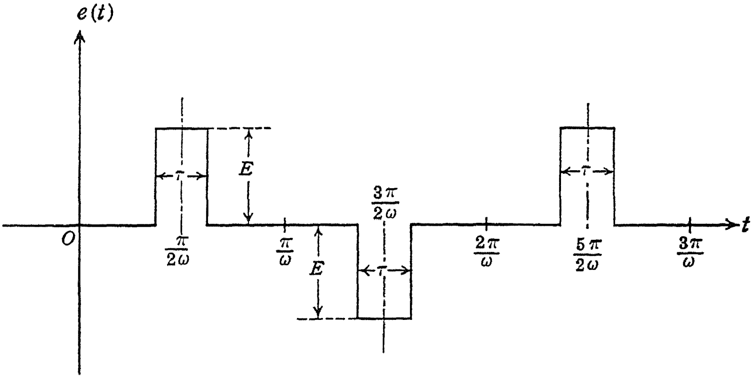

FIG. 139

course, idealizing the problem to some extent. Actually, the corners of these rectangular pulses will be considerably rounded off. But in idealizing the problem in this way, we nevertheless retain its essential characteristics, and at the same time make the solution very much simpler. Idealization is a common device which is used in the attack upon otherwise complicated problems. It is one of the objects of this presentation to introduce the reader to such methods. They may be frequently applied in a variety of forms.

We proceed with the expansion of the force function illustrated in the Fig. 139. By (974) we have:

$$a_\nu = \frac{\omega E}{2\pi}\left[\int_{\frac{\pi}{2\omega}-\frac{\tau}{2}}^{\frac{\pi}{2\omega}+\frac{\tau}{2}} e^{-j\nu\omega t}\,dt - \int_{\frac{3\pi}{2\omega}-\frac{\tau}{2}}^{\frac{3\pi}{2\omega}+\frac{\tau}{2}} e^{-j\nu\omega t}\,dt\right]. \tag{1026}$$

Here the second integral is identical with the first except for the limits of integration, which are advanced by $\frac{\pi}{\omega}$. Hence, we may combine the above into a single integral, thus:

$$a_\nu = \frac{\omega E}{2\pi}(1 - e^{-j\nu\pi})\int_{\frac{\pi}{2\omega}-\frac{\tau}{2}}^{\frac{\pi}{2\omega}+\frac{\tau}{2}} e^{-j\nu\omega t}\,dt. \tag{1027}$$

Evaluation gives:

$$a_\nu = \frac{\omega E(1 - e^{-j\nu\pi})e^{-j\nu\frac{\pi}{2}}}{2\pi j\nu\omega}\left(e^{j\nu\frac{\omega\tau}{2}} - e^{-j\nu\frac{\omega\tau}{2}}\right). \tag{1028}$$

The factor $(1 - e^{-j\nu\pi})$ makes the zero component and the even harmonics vanish. For odd integers of ν it is equal to 2. For these we also have:

$$e^{-j\nu\frac{\pi}{2}} = j(-1)^{\frac{\nu+1}{2}}. \tag{1029}$$

This is a factor which controls the algebraic sign of the harmonics. If we also introduce the notation:

$$\epsilon = \frac{\omega\tau}{\pi} \tag{1030}$$

then (1028) becomes:

$$a_\nu = (-1)^{\frac{\nu-1}{2}}\cdot\frac{2E\sin\nu\epsilon\frac{\pi}{2}}{j\nu\pi}. \tag{1031}$$

The new parameter ϵ is evidently the ratio of the duration of the pulse τ to the fundamental half-period. This is a very convenient parameter to use. It is a real number between zero and unity, depending upon the relative width of the pulse as compared to the fundamental half-period. $\epsilon = 1$ thus makes this function go over into the square wave treated before. Hence, the latter is a special case in the present problem. This is also evident from (1031). For $\epsilon = 1$:

$$\sin \nu\epsilon\frac{\pi}{2} = \sin \nu\frac{\pi}{2} = (-1)^{\frac{\nu-1}{2}} \tag{1032}$$

so that we get:

$$a_\nu = \frac{2\,E}{j\nu\pi}$$

which checks with the result (979*a*) for the square wave.

The process of solution for the R, L, C circuit with this force function impressed is exactly the same as for the square wave, i.e., the expression (1015) correctly gives the current coefficients where the voltage coefficients are now given by (1031). With the abbreviation (1016) we, therefore, have for the current in this case:

$$i = \sum_{\nu=-\infty}^{\infty} \frac{2\,E \sin \nu\epsilon\frac{\pi}{2}}{j\nu\pi\,(R + jX_\nu)} (-1)^{\frac{\nu-1}{2}} \cdot e^{j\nu\omega t} \tag{1033}$$

where the summation is to extend over odd integer values only.

Here the factor:

$$\sin \nu\epsilon\frac{\pi}{2}$$

is very interesting. For:

$$\nu\epsilon = \text{odd integer values} \tag{1034}$$

it has a magnitude of unity. For:

$$\nu\epsilon = \text{even integer values} \tag{1035}$$

it is zero. This means that those harmonics for which (1034) is satisfied, or nearly satisfied, will be favored as compared to the harmonics for which (1035) holds. Thus, we see that the duration of the pulse is a very important parameter in determining the harmonic selection process. In the case of the square wave, ϵ is unity, so that the relation (1034) is satisfied for all the harmonics because the even ones do not exist. Hence, in that case the selective properties of the system depended only upon the denominator of the expression (1033), i.e., principally upon the variation of X_ν. Now the selective properties are more critical. Hence, the pulse system of energizing an oscillatory circuit for the selection of harmonics, i.e., for frequency multiplication, leads to a purer resulting wave.

This is easily demonstrated. Consider the same numerical case as already treated and illustrated in Fig. 138, but with the pulse-wave of Fig. 139 as an exciting source, where we shall let:

$$\epsilon = \tfrac{1}{5}$$

because we wish to particularly favor the fifth harmonic. Then the relative amplitudes will be those given by (1021), multiplied by the factors:

$$\left.\begin{aligned}
\sin\frac{1}{5}\frac{\pi}{2} &= 0.31 & \sin\frac{3}{5}\frac{\pi}{2} &= 0.81\\
\sin\frac{5}{5}\frac{\pi}{2} &= 1.00 & \sin\frac{7}{5}\frac{\pi}{2} &= 0.81\\
\sin\frac{9}{5}\frac{\pi}{2} &= 0.31 & \sin\frac{11}{5}\frac{\pi}{2} &= 0.31\\
\sin\frac{13}{5}\frac{\pi}{2} &= 0.81 & \sin\frac{15}{5}\frac{\pi}{2} &= 1.00
\end{aligned}\right\} \qquad (1036)$$

respectively. This result is illustrated in Fig. 140. Comparing this figure with the Fig. 138, we see that the neighboring harmonics are somewhat suppressed, and those farther removed are very much suppressed. Hence, the impulsive excitation gives rise to a purer selected wave.

In this case it may also be interesting to determine the power absorbed by the load. For this purpose the effective value of the expression (1033) must be found. This is easily accomplished by means of the relation (1001) or (1001*a*). If we use the latter and remember that the summation extends only over the odd integer values, then we have for this case:

$$i_{\text{eff}}^2 = \sum_{\nu=1,3,5\ldots}^{\infty} \frac{4\,E^2 \sin^2 \nu\epsilon\frac{\pi}{2}}{\pi^2\nu^2(R^2 + X_\nu^2)}. \qquad (1037)$$

This summation may also be carried out, as will be shown later.

These examples should illustrate the superiority of the complex form over the usual sine and cosine form, by means of which these problems would have been very difficult to attack. In general, the solution to any network problem with a periodic force function is no harder to set up analytically than for a harmonic function. When the force function is impressed in mesh #s and the response is observed in mesh #k, then the latter will be in the form of a complex series whose coefficients are given by:

$$b_\nu = \frac{a_\nu}{Z_{sk}(\nu\omega)} \tag{1038}$$

where a_ν are the coefficients in the series representing the force function. Here the network impedance is the same as for any

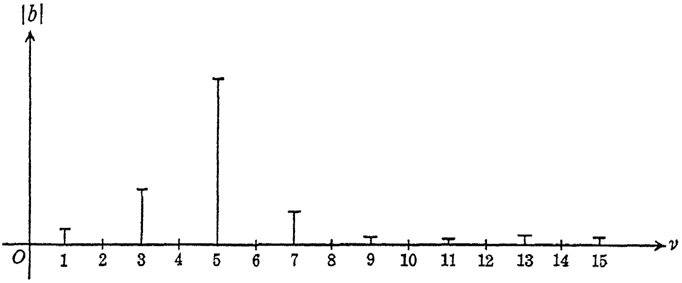

FIG. 140

steady-state problem with a harmonic force, except that it is a function of $\nu\omega$ instead of just ω. The amount of labor involved in carrying out the solution numerically is considerably more than for the simple harmonic case. In fact the determination of each harmonic amplitude requires the same mechanical motions as are needed for the determination of one steady-state solution.

5. Other uses for Periodic Functions. Fourier series are very often used in communication problems in other ways than those that have been taken up so far. The force function may be constant, and yet give rise to a non-harmonic current, when one or more of the circuit parameters vary with time. The commonest of these cases is the carbon microphone problem. The situation for the latter is illustrated in Fig. 141. It consists essentially of a constant source in series with the microphone and a constant resistance R_x across which the output voltage appears. The microphone represents a resistance which varies with time due to the impinging sound waves. This variation in the resistance of the circuit causes a varying current, and hence a varying voltage at the terminals of R_x. The object is to have this voltage variation be as nearly proportional to the resistance variation of the microphone as possible.

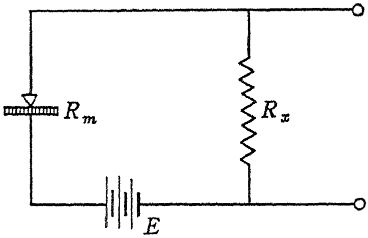

FIG. 141

This problem is usually analyzed by assuming that a harmonic sound produces a harmonic variation in the microphone resistance. It is then necessary to determine the current in the circuit. We shall attack this problem by letting the total resistance in the circuit be given by the Fourier series:

$$R(t) = R_m + R_x = \sum_{k=-\infty}^{\infty} R_k e^{jk\omega t}. \tag{1039}$$

The constant term R_0 will then take the place of the output resistance R_x plus the constant portion of the microphone resistance. The heuristic method suggests that the steady-state current is also a Fourier series with the same fundamental period, i.e., that:

$$i = \sum_{s=-\infty}^{\infty} b_s e^{js\omega t}. \tag{1040}$$

The equilibrium of forces then gives:

$$iR(t) = E$$

or:

$$\sum_{\substack{k=-\infty \\ s=-\infty}}^{\infty} R_k b_s e^{j(k+s)\omega t} = E. \tag{1041}$$

The solution is thus obtained by equating coefficients of the same harmonic frequency on both sides of this condition equation. On the right-hand side only the zero frequency term exists; the others are zero. If we, therefore, equate the coefficients of, say the rth harmonic on both sides, this becomes:

$$\sum_{k=-\infty}^{\infty} R_k b_{r-k} = E\delta_r \tag{1042}$$

where:

$$\left.\begin{aligned} \delta_r &= 1 \quad \text{for} \quad r = 0 \\ \delta_r &= 0 \quad \text{for} \quad r \neq 0. \end{aligned}\right\} \tag{1043}$$

The definition of the factor δ_r is merely a convenient device which aids in the analytic work. The left-hand side of (1042) is an infinite sum. This is as it should be because the sum in (1041) is a double one. Hence, an infinite number of terms in this double sum contribute to the coefficient of the rth harmonic, namely all those products of R_k and b_s for which:

$$k + s = r. \tag{1044}$$

For these:

$$s = r - k \tag{1045}$$

which accounts for the subscript of b in (1042).

Equation (1042) is the solution to our problem. It is in an extremely inconvenient form, because it represents an infinite system of simultaneous algebraic equations, each of which contains all the current harmonics as unknowns. The situation becomes somewhat relieved when the resistance variation is harmonic. This may be assumed without much loss in the utility of the solution, because a knowledge of the result for a harmonic variation of R_m of any frequency, is a considerable step in the right direction. This assumption limits the sum in (1042) to three terms, namely those for:

$$\left.\begin{aligned} k &= -1 \\ k &= 0 \\ k &= +1. \end{aligned}\right\}$$

For these we get:

$$R_{-1}b_{r+1} + R_0 b_r + R_1 b_{r-1} = E\delta_r. \tag{1046}$$

Since it makes no difference what the phase of the harmonic variation in R_m is, we may assume that:

$$R(t) = R_0 + 2\,R\cos\omega t. \tag{1047}$$

Then:

$$R_{-1} = R_1 = R \tag{1048}$$

because for this condition:

$$R_{-1}e^{-j\omega t} + R_1 e^{j\omega t} = 2\,R\cos\omega t.$$

With (1048) the eq. (1046) becomes:

$$b_{r-1} + \frac{R_0}{R} b_r + b_{r+1} = \frac{E}{R} \delta_r. \tag{1049}$$

If we let r take on any integer value from minus infinity to plus infinity, then (1049) gives an infinite set of equations with an infinite set of unknowns again. The reader should write this out to appreciate how the system is formed. He will see that each equation contains only three unknowns. This simplifies the situation considerably. Such a system may be solved in several ways. One method expresses the solution in the form of a repeating fraction. But such complicated methods are not necessary at all. Usually the series of current coefficients converges very rapidly, so that an approximate method is easily applied. This proceeds as follows.

In (1049) let $r = 0$. Then we have:

$$b_{-1} + \frac{R_0}{R} b_0 + b_1 = \frac{E}{R} \cdot \tag{1050}$$

Here the coefficient b_0 is usually by far the largest, so that we can write with sufficient accuracy:

$$R_0 b_0 = E$$

from which

$$b_0 = \frac{E}{R_0} \tag{1051}$$

i.e., the constant current term is approximately equal to the voltage divided by the constant part of the resistance. Now put $r = 1$ in (1049). This gives:

$$b_0 + \frac{R_0}{R} b_1 + b_2 = 0. \tag{1052}$$

Here b_2 is negligible compared to b_0 and b_1. Hence:

$$b_1 = -\frac{R}{R_0} b_0. \tag{1053}$$

In the same way we get:

$$b_2 = -\frac{R}{R_0} b_1 \tag{1054}$$

and so forth.

After an approximate set of coefficients have thus been determined, the relation (1049) may be applied over again to obtain a more accurate set of values. This process usually converges very rapidly. We shall illustrate with a numerical example. Suppose we have:

$$R(t) = 100 + 20 \cos \omega t. \tag{1055}$$

Then:

$$R = 10 \qquad R_0 = 100. \tag{1056}$$

Also let:

$$E = 100. \tag{1057}$$

We have approximately:

$$\left.\begin{aligned} b_0 &= 1 \\ b_1 &= -0.1 \\ b_2 &= 0.01 \\ b_3 &= -0.001 \\ b_4 &= 0.0001 \\ &\text{etc.} \end{aligned}\right\} \tag{1058}$$

Now since:

$$\left.\begin{aligned} b_{-1} &= b_1 \\ b_{-2} &= b_2 \\ &\text{etc.} \end{aligned}\right\}$$

we have by (1050):

$$b_0 = 1 + 0.2 = 1.2 \tag{1059}$$

as the second approximation to the constant component. For $r \neq 0$ we get from (1049):

$$b_r = -\frac{R}{R_0}(b_{r-1} + b_{r+1}). \tag{1060}$$

Applying this to the set (1058) and the corrected value (1059), we get:

$$\left.\begin{aligned} b_1 &= -0.1\,(1.2 + .01) = -0.121 \\ b_2 &= -0.1\,(-0.121 - .001) = 0.0122 \\ b_3 &= -0.1\,(.0122 + .0001) = -0.00123 \\ &\text{etc.} \end{aligned}\right\} \tag{1061}$$

By (1050) we then have:

$$b_0 = 1 + 0.242 = 1.242. \tag{1059a}$$

With this value and the set (1061), another corrected set is determined.

The method converges more rapidly the smaller R is compared to R_0. When R is equal to or greater than R_0, the method diverges and cannot be used. However, this is practically never the case. R_0 is usually very large compared to R. It must be in order to suppress the undesirable harmonics. When R_0 is one-hundred times as large as R or larger, then the solution is near enough to the correct value after the first set of approximations are calculated. That is, we may apply (1051) for the constant component of current, and calculate the harmonics by:

$$b_r = -\frac{R}{R_0} b_{r-1}. \tag{1062}$$

In particular the fundamental, which is the desired component, becomes:

$$b_1 = -\frac{R}{R_0} \cdot \frac{E}{R_0}. \tag{1063}$$

The second harmonic will be one per cent or less of the fundamental; the third harmonic will be one one-hundredth of one per cent or less of the fundamental, etc. The magnitude of the output voltage then is very nearly:

$$2\,|b_1| R_0 = 2\,E\frac{R}{R_0}. \tag{1064}$$

From this we see that it is necessary, from the standpoint of suppressing undesirable harmonics, to keep the output resistance high as compared to the variable microphone resistance. According to (1064), however, this reduces the output voltage of the device. Hence, it is not possible with this arrangement to combine a large output with good wave form.

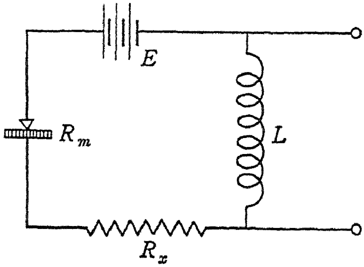

FIG. 142

The problem just treated involves the simplest possible microphone circuit. Very frequently the latter also contains inductance, at the terminals of which the output voltage appears. We represent this condition in Fig. 142. The treatment of this case is the same as for the previous one. We let the total resistance be given by the series (1039), and the current by (1040). Then we know that the equilibrium of forces is given by:

$$L\frac{di}{dt} + R(t) \cdot i = E. \tag{1065}$$

Substitution gives:

$$\sum_{s=-\infty}^{\infty} js\omega Lb_s e^{js\omega t} + \sum_{\substack{s=-\infty \\ k=-\infty}}^{\infty} R_k b_s e^{j(k+s)\omega t} = E. \tag{1065a}$$

Equating coefficients of like powers of the exponential, i.e., for the rth harmonic, we have:

$$jr\omega Lb_r + \sum_{k=-\infty}^{\infty} R_k b_{r-k} = E\delta_r \tag{1066}$$

where δ_r has the significance given by (1043). In particular we have for a harmonically varying microphone resistance:

$$R_1 b_{r-1} + (R_0 + jLr\omega)b_r + R_{-1}b_{r+1} = E\delta_r. \tag{1067}$$

This result is very similar to (1049) for the case without inductance. It again represents an infinite system of simultaneous equations. We may also look upon this result and the previous one given by (1049) as a recursion formula expressing any harmonic component in terms of the two preceding. That is, when two successive coefficients are known, any number of additional ones may be found by recursion. However, the accuracy of this method of determining successive coefficients rapidly diminishes. We, therefore, prefer to calculate them by the method of successive approximations given above.

Here we again assume (1047) for the resistance variation, and therefore have by (1067):

$$b_{r-1} + \alpha_r b_r + b_{r+1} = \frac{E}{R}\delta_r \tag{1068}$$

where:

$$\alpha_r = \frac{R_0 + jLr\omega}{R}. \tag{1069}$$

The results for this case are somewhat different. We will discuss the approximate solution only. For the constant component we neglect the harmonics, and obtain from (1068) for $r = 0$:

$$\alpha_0 b_0 \cong \frac{E}{R}$$

or:

$$b_0 \cong \frac{E}{R_0} \tag{1070}$$

as in the previous case. For the fundamental and higher harmonics we then have:

$$b_r = -\frac{b_{r-1}}{\alpha_r}. \tag{1071}$$

In many cases R_0 will be small compared to $Lr\omega$. Then:

$$\alpha_r \cong \frac{jLr\omega}{R} \tag{1072}$$

and (1071) becomes:

$$b_r = -\frac{Rb_{r-1}}{jLr\omega}. \tag{1073}$$

From this we see that the series of harmonic coefficients converge as the ratio:

$$\frac{R}{Lr\omega} \tag{1074}$$

which is smaller for the higner frequencies than it is for the low. Hence, the latter will be distorted more than the former. The convergence is in general better than for the previous case, however, on account of the index r in the denominator of (1073), and also because $L\omega$ is usually very large.

There is another very marked advantage that this circuit has as compared to the other. Here the output voltage can be taken as the drop across the inductance. Then it is given by:

$$2\,b_1 jL\omega = 2\,Rb_0 = \frac{2\,RE}{R_0}. \tag{1075}$$

Although this is the same result as (1064) for the circuit without inductance, the ratio of R to R_0 need not be made small in order to have the higher harmonics suppressed. These are suppressed by making the ratio (1074) small over the essential frequency range.

In the numerical example of the previous case we saw that the second harmonic was ten per cent of the fundamental. This is too high for good results. We will now calculate a numerical example for this case and compare it with the preceding. Suppose we have:

$$R = 10 \qquad R_0 = 100 \qquad L = 10 \qquad E = 100. \tag{1076}$$

Here R_0 is chiefly the resistance of the inductance. The above relation (1075) is still approximately correct because the inductive reactance will be so much larger than the resistance.

By (1070) the approximate constant component of current is the same as for the resistance case, namely:

$$b_0 = \frac{100}{100} = 1. \tag{1077}$$

For the fundamental frequency we will take the so-called average voice frequency, for which:

$$\omega = 5000. \tag{1078}$$

Then we have by (1073) for $r = 1$:

$$b_1 = \frac{jb_0}{5000} = \frac{j}{5000}. \tag{1079}$$

Hence by (1075) the magnitude of the fundamental output voltage becomes:

$$2\,|b_1|L\omega = 20 \text{ volts} \tag{1080}$$

which is the same as the approximate value for the preceding case, according to (1058) and (1064). However, by putting $r = 2$ in (1073), the second harmonic current is now:

$$b_2 = -\frac{1}{10000 \cdot 5000} \tag{1081}$$

so that the magnitude of the second harmonic output voltage becomes:

$$2\,|b_2|L2\,\omega = 4 \cdot 10^{-3} \text{ volts} \tag{1082}$$

which is 0.02 of one per cent of the fundamental output voltage (1080). In the resistance case the second harmonic output voltage was approximately ten per cent of the fundamental.

This comparison of the two cases may be summarized as follows, on the basis of the approximate solutions. For the case with resistance alone we have by (1062):

$$b_{r+1} = -\frac{R}{R_0}b_r = \frac{R^2}{R_0^2}b_{r-1}$$

and:

$$b_r = -\frac{R}{R_0}b_{r-1}.$$

The corresponding output voltages may be denoted by:

$$e_{r+1} = 2\,R_0|b_{r+1}|$$
$$e_r = 2\,R_0|b_r|$$

so that:

$$\frac{e_{r+1}}{e_r} = \frac{R}{R_0}. \tag{1083}$$

For the inductance case we have by (1073):

$$b_{r+1} = -\frac{Rb_r}{jL(r+1)\omega} = \frac{-R^2b_{r-1}}{L^2\omega^2r(r+1)}$$

and:

$$b_r = -\frac{Rb_{r-1}}{jL\omega r}.$$

Now:

$$e_{r+1} = 2\,L(r+1)\omega|b_{r+1}|$$
$$e_r = 2\,Lr\omega|b_r|$$

so that:

$$\frac{e_{r+1}}{e_r} = \frac{R}{Lr\omega}. \tag{1084}$$

The magnitude of the fundamental output voltage, according to (1064) and (1075), is given for both cases by:

$$e_1 = \frac{2\,RE}{R_0}. \tag{1085}$$

The superiority of the inductive case is due to the fact that the convergence may be controlled separately from the magnitude of the fundamental output. All this is true, of course, only in so far as the above approximations are justified, which in turn depends upon the relative magnitudes of the parameters involved.

It is interesting to compare these results for the carbon microphone with those for the condenser transmitter, which may be determined in a very similar manner. The usual circuit for the condenser transmitter is given in Fig. 143. The variable elastance of the condenser transmitter is denoted by $S(t)$. The output voltage is taken from the terminals of the resistance R.

This circuit is more readily analyzed in terms of the mesh charge, which we will assume to be given by the series:

$$q = \sum_{s=-\infty}^{\infty} b_s e^{js\omega t}. \tag{1086}$$

The variable elastance may be represented by:

$$S(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\omega t}. \tag{1087}$$

The equilibrium of forces demands that:

$$R\frac{dq}{dt} + S(t) \cdot q = E. \tag{1088}$$

This equation has the same form as (1065), which is the corresponding one for the microphone circuit with inductance. The charge replaces the current; the resistance replaces the inductance; and the variable elastance replaces the variable resistance. With the same relative magnitudes of the parameters, the results for the condenser transmitter must, therefore, be identical with those for the microphone, neglecting other factors not explicitly involved in this analysis, of course.

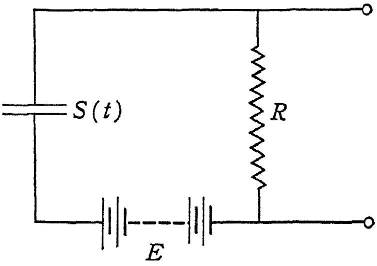

FIG. 143

Substitution of (1086) and (1087) into (1088), gives:

$$R\sum_{s=-\infty}^{\infty} js\omega b_s e^{js\omega t} + \sum_{\substack{k=-\infty \\ s=-\infty}}^{\infty} S_k b_s e^{j(k+s)\omega t} = E. \tag{1089}$$

Equating coefficients of like harmonics, we have:

$$jr\omega R b_r + \sum_{k=-\infty}^{\infty} S_k b_{r-k} = E\delta_r \tag{1090}$$

where δ_r has the significance expressed by (1043) again. Assuming a harmonic variation in $S(t)$, i.e.:

$$S(t) = S_0 + 2\,S\cos\omega t \tag{1091}$$

we have by (1090):

$$b_{r-1} + \alpha_r b_r + b_{r+1} = \frac{E}{S}\,\delta_r \tag{1092}$$

with:

$$\alpha_r = \frac{S_0 + jRr\omega}{S} \tag{1093}$$

where:

$$S = S_1 = S_{-1}. \tag{1094}$$

The harmonic variation in the elastance of the condenser transmitter indicated by (1091) assumes that a harmonic sound, moves the condenser plates with harmonic motion. Whether this is true or not will not be discussed here. The chief difference between this case and the microphone circuit with inductance is that, practically it is quite easy in the latter to make $L\omega$ large compared to R_0 over the essential frequency range, whereas it is not so simple to make $R\omega$ large compared to S_0 in the condenser transmitter circuit. This means that the low frequencies may be somewhat suppressed as compared to the high frequencies. The reason for this will become clear in the following.

Approximately, we have from (1092) for $r = 0$:

$$\alpha_0 b_0 = \frac{E}{S}$$

or:

$$b_0 = \frac{E}{S_0}\cdot \tag{1095}$$

For the fundamental and higher harmonics we may neglect the term for $(r + 1)$ in (1092) and get:

$$b_r = -\frac{b_{r-1}}{\alpha_r}. \tag{1096}$$

The magnitude of the output voltage is given by:

$$e_r = 2|b_r|Rr\omega$$

or:

$$e_r = 2\,Rr\omega\left|\frac{b_{r-1}}{\alpha_r}\right|\cdot \tag{1097}$$

The convergence of harmonics is given by:

$$\frac{e_r}{e_{r-1}} = \frac{1}{|\alpha_r|} \cdot \frac{r}{r-1} \tag{1098}$$

which is found from (1097). The fundamental, or useful output is:

$$e_1 = \frac{2\,SR\omega E}{S_0|S_0 + jR\omega|}. \tag{1099}$$

Consider the following numerical data:

$$\left.\begin{aligned} S_0 &= 3 \cdot 10^9 \\ S &= 3 \cdot 10^5 \\ R &= 10^7 \end{aligned}\right\} \tag{1100}$$

which is fairly representative. Then:

$$e_1 = \frac{2\,\omega \cdot 10^{-4} \cdot E}{|300 + j\omega|}. \tag{1101}$$

This result is plotted in Fig. 144. It shows that the output drops off at low frequencies. This is also the case for the microphone circuit with inductance; but there the curve does not drop until very low frequencies are reached because $L\omega$ is usually larger compared to R_0 than $R\omega$ is compared to S_0. The condenser transmitter circuit requires very high values of R to bring out the low frequencies properly, because S_0 is so large on account of the inherently small capacity of the transmitter. However, when it comes to suppressing undesirable harmonics, the condenser transmitter circuit is much better than the microphone circuit. In the

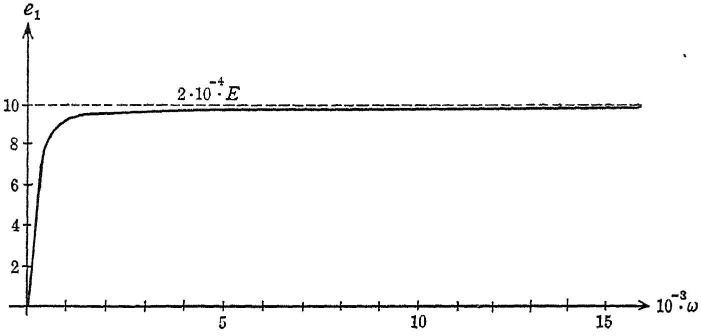

FIG. 144

last numerical example for the latter we found by (1082) that the second harmonic voltage was 0.02 of one per cent of the fundamental. For the condenser transmitter we have by (1098) and (1078):

$$\frac{e_2}{e_1} = \frac{6 \cdot 10^5}{|3 \cdot 10^9 + j\,10^{11}|} \cong 6 \cdot 10^{-4}\,\% \qquad (1102)$$

which is practically nothing at all.

There are many other problems that can be treated in a similar manner to the above. The general method of attack is always the same. A complex series is assumed for the current and whatever else varies periodically with time, and the equilibrium of forces is then expressed in terms of these series. By equating coefficients of like harmonics we then obtain some relation between the unknown current coefficients, the forces, and the known parameters. This relation will usually be in the form of an infinite system of simultaneous equations as in the above examples. As pointed out above, this infinite system may also be looked upon as a recursion formula. The solution in any case is usually best found by a judicious approximation method.

6. The Development of Summation Formulae. In applying Fourier series to network problems, we very frequently arrive at solutions in the form of infinite series such as are given by the formulae (1025) and (1037), for example. In such cases it would be more convenient for purposes of calculation to have the solution in a closed form. This may be arrived at by carrying out the indicated summation. For this purpose we must develop suitable formulae. The latter may be obtained by an ingeneous use of

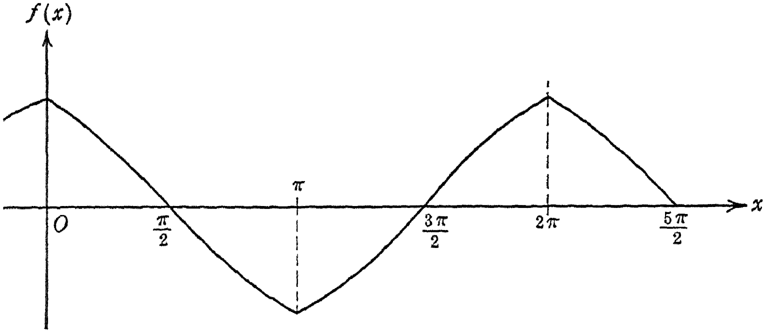

FIG. 145

Fourier series which is not generally known. The writer was first introduced to this subject some years ago through the kindness of Professor A. Sommerfeld, to whom he also owes much in connection with the use of complex series in general. A brief outline of the outstanding features of this method will now be given.

In order to develop a suitable summation formula for our purpose, we consider the function illustrated in Fig. 145. This function is defined by:

$$\left.\begin{array}{ll} f(x) = \sin m\left(\frac{\pi}{2} - x\right) & 0 < x < \pi \\ f(x) = -\sin m\left(\frac{\pi}{2} - x\right) & \pi < x < 2\,\pi \\ \text{etc.} & \end{array}\right\} \tag{1103}$$

The factor m is here taken to be a real number less than unity. This is merely done to be able to plot the function, and thus give the reader a better idea of what we are doing. Later we shall not restrict m in any way.

The Fourier expansion of this function is determined in the usual fashion and found to be the following cosine series:

$$f(x) = \sum_{\nu=1,3,5\ldots}^{\infty} \frac{4\,m \cos m\frac{\pi}{2}}{\pi(\nu^2 - m^2)} \cos \nu x. \tag{1104}$$

For values of x between zero and π we, therefore, have:

$$\sum_{\nu=1,3,5\ldots}^{\infty} \frac{\cos \nu x}{\nu^2 - m^2} = \frac{\pi \sin m\left(\frac{\pi}{2} - x\right)}{4\,m \cos m\frac{\pi}{2}}. \tag{1105}$$

This is the first form of summation formula that we arrive at in this fashion. It is valid for any value of x (inclusive of the boundaries) within the above specified region. The value of m may be real or complex. In particular for $x = 0$, we get:

$$\sum_{\nu=1,3,5\ldots}^{\infty} \frac{1}{\nu^2 - m^2} = \frac{\pi \sin m\frac{\pi}{2}}{4\,m \cos m\frac{\pi}{2}} = \frac{\pi}{4\,m} \tan m\frac{\pi}{2}. \tag{1106}$$

It is interesting in passing to note that this gives:

$$\tan m\frac{\pi}{2} = \frac{1}{\pi}\sum_{\nu=1,3,5\ldots}^{\infty} \frac{4m}{\nu^2 - m^2} \tag{1107}$$

which is the partial fraction expansion of the tangent function.[1]

The further development of summation formulae now proceeds by various manipulations of the forms (1105) and (1106). For example, since by trigonometry:

$$1 - \cos \nu x = 2 \sin^2 \nu\frac{x}{2} \tag{1108}$$

we get by subtracting (1105) from (1106):

$$\sum_{\nu=1,3,5\ldots}^{\infty} \frac{2\sin^2 \nu\frac{x}{2}}{\nu^2 - m^2} = \frac{\pi\left\{\sin m\frac{\pi}{2} - \sin m\left(\frac{\pi}{2} - x\right)\right\}}{4m \cos m\frac{\pi}{2}}. \tag{1109}$$

This begins to look something like the form required for the summation of (1037). We need to perform one more transformation of (1109), however, before we arrive at the proper form. For the sake of abbreviation let:

$$f(m) = \frac{\pi\left\{\sin m\frac{\pi}{2} - \sin m\left(\frac{\pi}{2} - x\right)\right\}}{4m \cos m\frac{\pi}{2}} \tag{1110}$$

Now consider the difference:

$$f(m) - f(\overline{m}) = 2\left(m^2 - \overline{m}^2\right) \sum_{\nu=1,3,5\ldots}^{\infty} \frac{\sin^2 \nu\frac{x}{2}}{\left(\nu^2 - m^2\right)\left(\nu^2 - \overline{m}^2\right)} \tag{1111}$$

which is obtained from (1109) by considering the two values m and $\overline{m}$, where the latter may be any other value. Finally, we introduce the change of variable:

$$x = \pi\epsilon \tag{1112}$$

and have:

$$\sum_{\nu=1,3,5\ldots}^{\infty} \frac{\sin^2 \nu\epsilon\frac{\pi}{2}}{\left(\nu^2 - m^2\right)\left(\nu^2 - \overline{m}^2\right)} = \frac{f(m) - f(\overline{m})}{2\left(m^2 - \overline{m}^2\right)}. \tag{1113}$$

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[1] For an interesting application of this form see A. Sommerfeld, "Zur Theorie der Lichtbogenschwingungen bei Wechselstrombetrieb," Sitzungsberichte d. K. B. Akad. d. Wiss., Math. Phys. Klasse, 1914.

This formula is capable of handling the form (1037), as we shall now show. Incidentally, we note by (1112) that (1113) will be valid for the interval:

$$0 \lesseqgtr \epsilon \lesseqgtr 1 \tag{1114}$$

which just fits our case.

Getting back to the expression (1037) for the effective current in an R, L, C circuit with impulsive excitation, we see by (1016) that:

$$R + jX_\nu = R + jL\nu\omega + \frac{1}{jC\nu\omega}\cdot \tag{1115}$$

By means of the decrement and natural angular velocity of this circuit:

$$\left.\begin{aligned} \alpha &= \frac{R}{2\,L} \\ g &= \sqrt{g_0^2 - \alpha^2} \\ g_0^2 &= \frac{1}{LC} \end{aligned}\right\} \tag{1116}$$

we note that (1115) may be factored, just like the determinantal equation for this circuit could be factored. We get:

$$R + jX_\nu = j\frac{L\omega}{\nu}(\nu - m)(\nu - \overline{m}) \tag{1117}$$

where:

$$\left.\begin{aligned} m &= \frac{g + j\alpha}{\omega} \\ \overline{m} &= \frac{g - j\alpha}{\omega}\,. \end{aligned}\right\} \tag{1118}$$

Note that m and $\overline{m}$ are conjugates.

From this it is easy to see that:

$$\left.\begin{aligned} (R + jX_\nu)(R - jX_\nu) &= R^2 + X_\nu^2 \\ &= \frac{L^2\omega^2}{\nu^2}(\nu^2 - m^2)(\nu^2 - \overline{m}^2). \end{aligned}\right\} \tag{1119}$$

By means of this result, the expression (1037) becomes:

$$i_{\text{eff}}^2 = \sum_{\nu=1,3,5\ldots}^{\infty} \frac{4\,E^2 \sin^2 \nu\epsilon\dfrac{\pi}{2}}{\pi^2L^2\omega^2(\nu^2 - m^2)(\nu^2 - \overline{m}^2)} \tag{1120}$$

which exactly fits the summation formula (1113).

By (1118) we see that:

$$m^2 - \overline{m}^2 = \frac{2\,Rg}{L\omega^2} \tag{1121}$$

so that (1120) becomes:

$$i_{\text{eff}}^2 = \frac{E^2}{\pi RLg}\mathcal{J}_m\left\{\frac{\sin m\dfrac{\pi}{2} - \sin m\dfrac{\pi}{2}(1 - 2\,\epsilon)}{m\cos m\dfrac{\pi}{2}}\right\}. \tag{1122}$$

This is the desired closed form for the effective current in an R, L, C circuit with the impulsive excitation illustrated in Fig. 139. The width of each impulse is controlled by the parameter ϵ, as pointed out before.

The result for the square wave of Fig. 135 is obtained by putting $\epsilon = 1$ in (1122). Hence, the expression (1025) for the power absorbed in this case becomes:

$$P = \frac{2\,E^2}{\pi\,Lg}\mathcal{J}_m\left\{\frac{\tan m\dfrac{\pi}{2}}{m}\right\} \tag{1123}$$

which is quite simple.

The same principle may be applied to develop summation formulae of a variety of forms. The difference-factors in the denominators of these formulae make them admirably suited for network analysis, because impedance functions can usually be factored so as to assume this form. We shall show how the general idea may be extended to obtain new forms.

For example, if we let $m = 0$ in (1105), we have:

$$\sum_{\nu=1,3,5\,\ldots}^{\infty} \frac{\cos\nu x}{\nu^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - x\right) \tag{1124}$$

which is a new form of summation formula. Putting $x = 0$ in the above, we have:

$$\sum_{\nu=1,3,5\,\ldots}^{\infty} \frac{1}{\nu^2} = \frac{\pi^2}{8}. \tag{1125}$$

This is also a well-known formula. From (1111) we can get another form by letting one of the m's go to zero, thus:

$$\sum_{\nu=1,3,5\,\ldots}^{\infty} \frac{\sin^2\nu\dfrac{x}{2}}{\nu^2(\nu^2 - m^2)} = \frac{4\,f(m) - \pi x}{8\,m^2}. \tag{1126}$$

Other forms may be obtained by differentiation. If we differentiate (1126) with respect to x, for example, we get:

$$\sum_{\nu=1,3,5\ldots}^{\infty} \frac{\sin \nu x}{\nu(\nu^2 - m^2)} = \frac{\pi \cos m\left(\frac{\pi}{2} - x\right) - \pi \cos m\frac{\pi}{2}}{4\, m^2 \cos m\frac{\pi}{2}}. \quad (1127)$$

In differentiating these forms in this way we must observe that this may be done only so long as the resulting series still converges as one over ν^2. If the latter is differentiated once more, then it will converge only as one over ν, which means that the function it represents has discontinuities at the boundaries:

$$x = 0$$

and:

$$x = \pi.$$

On account of the Gibbs phenomenon, the series will then no longer be valid at these boundaries. For example, if we differentiate (1124) with respect to x, we have:

$$\sum_{\nu=1,3,5\ldots}^{\infty} \frac{\sin \nu x}{\nu} = \frac{\pi}{4}$$

which is obviously incorrect at the boundaries.

Otherwise we may perform additions and subtractions, carry out various limits, and differentiate and integrate the forms in order to obtain almost any kind of summation formula which falls into this class.

PROBLEMS TO CHAPTER X

10–1. An inductance has impressed upon it a square wave of voltage. Determine and plot the shape of the resulting steady-state current.

10–2. A condenser has impressed upon it a triangular voltage wave. Determine and plot the shape of the resulting steady-state current. What will the steady-state charge look like?

10–3. The diagram of connections for a double-button carbon microphone transmitter is essentially a " push-pull " circuit. Show that this arrangement may be used to eliminate the even harmonics in the output.

10–4. Solve the condenser transmitter circuit of Fig. 143 with the resistance replaced by an inductance. From the form of this solution state why this circuit cannot be used for transmitting intelligence.

10–5. Consider a square voltage wave with a fundamental period of $\frac{\pi}{\sqrt{2}} \cdot 10^{-3}$ seconds impressed in mesh #1 of the network defined by Prob. **8–10** with the additional parameters of Prob. **8–13.** Determine the resulting selective properties in mesh #2 of this network, and plot the relative magnitudes of the various harmonics in the manner illustrated by Fig. 138.

10–6. Repeat Prob. **10–5** using the impulsive form of excitation illustrated by Fig. 139, with the same fundamental period and $\epsilon = 0.2$.

10–7. Repeat Prob. **10–5** for a fundamental period of:

$$\frac{\pi}{\sqrt{6}} \cdot 10^{-3} \text{ seconds.}$$

10–8. Repeat Prob. **10–6** with the fundamental period given by Prob. **10–7.**

10–9. For the network of Fig. 114 consider the generator to have a square wave form with a fundamental period equal to $\pi \cdot 10^{-4}$ seconds. The network parameters are given by:

$$R = 10 \qquad L = 10^{-2} \qquad C = 10^{-8} \qquad M = 2 \cdot 10^{-4}.$$

Determine the magnitudes of the current harmonics in the second mesh, and plot in the manner illustrated in Fig. 138.

10–10. Repeat Prob. **10–9** with the impulsive form of excitation illustrated in Fig. 139, with the same fundamental period and $\epsilon = 0.2$. Compare with the results of Prob. **10–9.**

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